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Research Reports
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Editors

Jeong-Ho Woo Hee-Chan Lew
Kyo-Sik Park Dong-Yeop Seo

The Korea Society of Educational Studies in Mathematics

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International Group for the Psychology of Mathematics Education*

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The Korea Society of Educational Studies in Mathematics
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HOW DO YOUR STUDENTS THINK ABOUT PROOF? A DVD RESOURCE FOR MATHEMATICIANS

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This paper is about the construction and initial testing of a DVD resource designed to help mathematicians learn more about their students' reasoning and engage with issues from mathematics education. The DVD uses specially annotated video data, together with screens of prompts for reflection, to encourage discussion among mathematicians about students' difficulties and successes. In the paper I describe part of the content in detail, showing how a short episode can raise many major issues in the learning of proof. I also report on design issues in the structure of the DVD, and on mathematicians' responses to an early version of the content.

INTRODUCTION: DVD PURPOSE AND STRUCTURE

Few mathematicians have time to read education research papers, so if we want them to be able to benefit from the ideas and results of mathematics education, we need to find ways to make these more accessible. This paper reports on a project that aims to do this by providing a DVD resource in which mathematicians can watch individual students engage with proof-based mathematics problems¹. Video episodes have been edited so that half of the screen shows the student working and the other half shows subtitles at the bottom in blue, and the student's written work in large print in black at the top (a screen shot is provided in Figure 1). Both the subtitles and written work change in real time, enabling the viewer to follow the student's thinking as easily as possible. The content of each episode is divided into 2-3-minute segments, after each of which the viewer sees a screen of questions designed to prompt reflection on what has been seen. In this way, mathematicians are invited to analyse this data for themselves, attempting to accurately characterise what they see and debating the learning and teaching issues that arise. They are thus able to engage with the process of mathematics education, rather than to passively consume its products. For another project with a similar ethos, see Nardi & Iannone (2004).

In this paper I describe one video episode, which has now been shown in presentations to four groups of mathematicians in the UK and the USA. In this episode, Nick attempts a problem about upper bounds. In this paper I demonstrate how a single short episode can raise numerous issues in mathematics education at the transition-to-proof level, including:

¹ This project is a Higher Education Academy Maths, Stats & OR Network funded activity.

- Students' understanding of abstract mathematical concepts and the way in which these must relate to definitions (*cf.* Vinner, 1991; Moore, 1994).
- Students' use of examples in their mathematical reasoning (*cf.* Bills et. al., 2006; Dahlberg & Housman, 1997).
- Students' strategies for proving and the extent to which they apply these effectively (*cf.* Alcock & Weber, in press; Weber, 2001).
- Students' understanding of what constitutes a proof, and the relationship between counterexamples and proof (*cf.* Recio & Godino, 2001; Iannone & Nardi, 2004).



2. uv is an UB of $A \cup B$
 uv UB A and uv UB B
 $uv \geq u$ $uv \geq v$
Not true Not true
If v is neg if u is negative

So...there's many other cases, but if - all you need to do is find one example, so that would be my one example.

Figure 1: Screen shot from the DVD.

DATA: DVD CONTENT

The data used for the DVD was gathered at a large state university in the USA. The participants were all enrolled in a transition-to-proof course called "Introduction to mathematical reasoning," in which they studied methods of proof in the context of abstract topics such as sets, functions and number theory. Each participant was interviewed individually, and during the interview was asked to attempt three tasks.

Each task was given on a piece of paper; the student was told they could write on the paper and was asked to describe their thinking out loud. The interviewer remained silent except to remind the participant of this request and to clarify the nature of the task. In this paper we see Nick working on the third of the three tasks:

Let $A \subseteq \mathbf{R}$ be a nonempty set. U is an **upper bound** for A if and only if $\forall a \in A$, $a \leq u$.

Task: Suppose that $A, B \subseteq \mathbf{R}$ are nonempty sets, u is an upper bound for A and v is an upper bound for B . Prove or disprove:

1. $u + v$ is an upper bound for $A \cup B$.
2. uv is an upper bound for $A \cup B$.
3. $u - v$ is an upper bound for $A \cap B$.

The reader will note that all three statements are false, so that in each case a “disproof” is required and hence a counterexample is needed. This makes the task different from those that most students will encounter in such courses. This was a deliberate research design choice, since I wanted to avoid questions that could be answered using a standard procedure.

The data was originally intended for research purposes only, so the content that could be used for creating the DVD was constrained by the limited number of students who agreed to allow their interviews to be used for this purpose. Fortunately, two of these in particular (Nick included) were articulate in expressing their thinking and made good progress on the interview tasks, without producing perfect solutions and without being maximally efficient. In my view this type of data is particularly suited to this purpose, because it can provide valuable insight into what might reasonably be expected from a competent student and what we might therefore realistically hope to teach our students to do at this level.

RESULTS: NICK’S ATTEMPT AND MATHEMATICIANS’ RESPONSES

Here I describe the four segments of video content in which Nick works on the upper bounds task, and give a short overview of the issues each raised for the mathematicians who have seen presentations of this material.

Segment 1: An incorrect general argument

Nick began by reading the task and writing the following symbolic summary of the given information. He then paused for several seconds and then announced his thoughts as follows:

In my head I’m thinking, alright, if u is the upper bound for A , and v is the upper bound for B ...whichever...set has the higher number...pretty much, that’s going to be the upper bound. So the upper bound for $A \cup B$ is either going to be u or v , so the sum of them will be the upper bound.

Nick next stated that he now needed to “figure out a way to prove this”, and after a considerable pause went on to consider two cases: u is greater than v and vice versa. He then explained that

...for the case where u is less than v , the upper bound would just be v . And in the other case the upper bound would just be u . Now it says for like both cases – the upper bound just means that it’s greater than any number in that set. So, if it’s u plus v that also means it could be...say... $3u$ plus $3v$. [...] So, if either one of these is an upper bound, I would say the sum of them is an upper bound. Because v plus u is greater than v . Because u cannot be zero because it’s not a...non-empty set. Or it can be zero so we’ll say greater than or equal to.

He then made the similar argument for the other case, and finished this first attempt by saying, “So, I would guess that would prove that v plus u is an upper bound.”

One common observation about this segment was that Nick repeatedly used the phrase “the upper bound”; mathematicians noted that he appeared to be confusing the idea of upper bound with that of either supremum or maximum element, which raised issues about conceptual understanding and the role of definitions in mathematics (cf. Moore, 1994; Vinner, 1991). Another observation was that his argument does not work for negative upper bounds. This raised issues about students’ understanding of the range of examples to which a statement might apply (cf. Bills et.al., 2006; Watson & Mason, 2002).

Segment 2: Consideration of negative numbers

Nick moved on to question 2 in the task and, after reading the question and pausing, he said,

Those could be negatives too.... What if [...] what if B has a maximum of a negative number?

He then reconsidered his first answer, observing that since A and B are subsets of the reals, either u or v could be negative. He then made the following comment about the overall task structure:

It says “prove or disprove”. I didn’t actually – I thought it said “prove”. My teacher never actually gives us a “disprove”.

Having established this, he went on to reason as follows.

If the set B is all negative numbers...that means its upper bound could be a negative number. So u plus v would be less than u . So it would not necessarily be greater. [...] And, pretty much, this [*indicating his earlier statement that $u + v \geq u$*] is not true...if u or v are negative.

He then offered the following comment on his own reasoning:

I don’t know how good a proof that is. But we’ll continue and go back to it if I think of anything else.

One common observation about this segment was that Nick had now resolved the issue of negative upper bounds, though there still seemed to be some confusion about the precise meaning of the term. Another was that he had not provided a specific counterexample, though he appeared to understand what properties this would need to have. This raised issues about students' understanding of what is required for a proof (*cf.* Recio & Godino, 2001) and the nature and use of counterexamples (*cf.* Iannone & Nardi, 2004). A third was that Nick's expectations of the task itself seemed to have had quite an impact on his first attempt at question 1. This raised issues about potential alternatives to the standard form of mathematical tasks, an in particular tasks that required some form of example generation (*cf.* Watson & Mason, 2002).

Segment 3: Using specific examples

Nick next re-read part 2 of the task and then said,

Well for the same reason, that's false. If u is negative and v is positive...then...the upper bound could be a negative number, and that's not going to be an upper bound for...the one with positive values.

He then hesitated before writing the following, noting that uv being an upper bound of $A \cup B$ would mean that uv was an upper bound of both A and B .

uv is an UB of $A \cup B$

uv UB A and uv UB B

$$uv \stackrel{?}{\geq} u \qquad uv \stackrel{?}{\geq} v$$

Having written this down, he began to restate his suggestion that v could be negative if the set B contained all negative values. He then interrupted himself to give the following comment on the structure of his argument:

Pretty much, to prove something – to disprove something you'd find an example, or let something exist. So, if B is a negative set – or a set of all, of only negative numbers, that means its upper bound could be a negative number. Meaning that this [*indicating* $uv \stackrel{?}{\geq} u$] is not true if v is negative.

He then commented,

So, there's many other cases, but if – all you need to do is find one example, so that would be my one example.

Observations about this segment again focused on the fact that although Nick had given a correct condition under which upper bounds of the sets would not be related as in the question, and although he had mentioned the strategy of giving an example, he still had not done so. This raised issues about the extent to which students are inclined to introduce examples and are able to use these effectively to assist them in their reasoning (*cf.* Alcock & Weber, in press; Dahlberg & Housman, 1997; Weber, Alcock & Radu, 2005).

Episode 4: Considering the role of counterexamples

Nick then moved on to the final part of the question, and this time he did introduce a specific example. He said,

If A is a set...this could be any set. I'll go 2...7. 2 through 7. And B is the set of...negative 5 to 2. Then we're saying that u ...in this case it would be 7 and 2. Saying that 5 is the upper bound of A and B . Of the intersection. That works in this case...mainly because...but what if it was...? Let's make them closer together. If A is 2 through 7 again, and B is...2 through 6. That would mean that, u minus v is 7 minus 6. It doesn't have to be – those are just examples of it. Trying to find an example where it doesn't work. And you'd get 1, which is not an upper bound. Of $A \cap B$. Because $A \cap B$ equals 2 through 6. $A \cap B$ is just B . So this...I've found an example where it doesn't work. So that would, pretty much...I have my way of proving that. I have to pretty much write the proof itself because.... Actually...

He then asked the interviewer,

Do I have to write the proof, if I can find an example? To disprove it?

The interviewer said that she would not answer questions of that nature at this point in the interview. After a good-natured groan, Nick then went on to look back at his first page. He hesitated and then said,

I was always told that to disprove something, all you need is an example where it doesn't work. So I'm just going to have this example...where u equals 7, v equals 6, u minus v equals 1. Which is not an upper bound of $A \cap B$, which is 2 through 6.

Having written this down, Nick then turned back to the previous questions again and said,

Actually now thinking that...all you need is an example...I really did not need to go through any of this. All I had to do was write an example for every one.

At this point he smiled and gave a good-natured curse, and then went on to provide examples for the previous parts of the question. For both parts he wrote $A = \{(2, 7)\}$, $B = \{(-10, -3)\}$ (note his misuse of set notation; once again he said "through," indicating that he meant the set of numbers between 2 and 7 etc.). After writing down these examples and calculating $u + v$ and uv respectively, he said,

So all you need to do is find one. And there's my one, so now I'm done with this one.

By this point, Nick did sound confident that this "give a single counterexample" strategy achieved what was required. So it was interesting that this part of the interview then concluded with the following exchange:

N: Now can you answer the question for me?

I: What was the question again?

N: Can you use an example to disprove something?

I: Yes.

N: Alright.

It is worth noting, however, that by this stage he did not seem surprised by the interviewer's response.

Observations about this part of the interview tended to focus on Nick's use of examples to support his mathematical reasoning (*cf.* Alcock & Weber, in press; Dahlberg & Housman, 1997; Weber, Alcock & Radu, 2005) and on the apparent developments in his understanding of the relationship between counterexamples and proof (*cf.* Recio & Godino, 2001; Iannone & Nardi, 2004).

DISCUSSION

Design of the DVD

The first thing to note is that the divided-screen format was extremely successful in allowing viewers to follow the students' reasoning. No-one in any of the presentations felt any need to remark upon it, and those who were specifically asked all said that they could follow it perfectly. A second point is that I realised while presenting the material that by splitting it into the obvious segments, I had unintentionally created "cliff-hangers": each segment ends with some kind of error or ambiguity in Nick's thinking, which he then resolves to some extent in the next segment. This is obviously not a necessary feature for such a presentation, but with hindsight it proved effective in two ways. First, having unresolved errors or ambiguities at breaks in the video seemed to provide the impetus for serious debates among the mathematicians about the precise nature of the errors, the possible sources of these and the way in which such errors might be resolved or avoided. Second, it gave the presentations a sense of drama – for most viewers there seemed to be strong sense of resolution and relief at various points, in particular when Nick first did begin to consider negative numbers. Given my goal of making a resource that would be engaging and enjoyable, this is obviously a plus.

What the mathematicians learned

I was pleased to find that the video-based presentation generated much discussion about the issues noted above. Many of the mathematicians present were keen to continue the discussions with me and with each other, and to hear more about the ways in which these issues are conceptualised within the mathematics education community. There were also three overarching themes that regularly occurred in summary discussion sessions, which I outline briefly here.

First, there was discussion of the general difficulty of discerning a student's thinking from their written work. Many of those who saw the presentation noted that what Nick said often indicated a greater understanding than what he wrote down. Second, there was reflection upon the mathematicians' own role in both task design and direct teaching situations. Many people noted that Nick's capacity for self-correction seemed to be higher than they might ordinarily allow for – that they would have interrupted him as he made his errors, but that having seen this video they would think twice about doing so in similar situations in future. Finally, a common remark at the end of the video was that Nick was quite a strong student, and that his thought processes were

obviously not perfect but were, in fact, almost exactly like those used by mathematicians in their own work. They noted that his errors, struggles, explorations, moments of insight and moments of uncertainty were very much like their own, and they accorded him considerable respect for his reasoning. In this, at least some had moved on considerably from initial amusement or horror at his errors and impatience with his slowness to invoke suitable strategies. Such respect for a student's thinking is surely a good basis for further serious engagement in issues in mathematics education.

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TEACHERS' CONCEPTIONS OF MATHEMATICAL CHALLENGE IN SCHOOL MATHEMATICS

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This study arose from our belief that mathematics should be challenging in any mathematics classroom. We analyse conceptions of mathematical challenge of two groups of experienced mathematics teachers. The first group was asked to define the notion of mathematical challenge and give examples of challenging mathematical tasks ($N_1=9$). The second group of teachers was presented with a questionnaire based on the replies of the teachers from the first group ($N_2=41$). We found that the teachers have a broad conception of mathematical challenge, appreciate relativity of mathematical challenge, but are not always convinced of the possibility of incorporating challenging mathematics in everyday classroom.

BACKGROUND

The role of challenge in mathematics education

This study is inspired by our participation in ICMI-16 study group "Challenging mathematics in and beyond the classroom". Peter Taylor (2006) in his presentation of the agenda of the conference wrote: "Challenge is not only an important component of the learning process but also a vital skill for life. People are confronted with challenging situations each day and need to deal with them. Fortunately the processes in solving mathematics challenges (abstract or otherwise) involve certain types of reasoning which generalise to solving challenges encountered in every day life".

In Cambridge Advanced Learner's Dictionary 'challenge' is defined as 'difficult job', something needing great mental (or physical) effort in order to be done successfully. Incorporation of a mathematical challenge in learning/teaching process involves both psychological and didactical considerations.

Principles of 'developing education' (Davydov, 1996), which integrate Vygotsky's (1978) notion of ZPD (Zone of Proximal Development) claim that to develop students mathematical reasoning the tasks should not be too easy or too difficult and the learners have to approach any task through meaningful activity. Another perspective on mathematical challenge may be seen in differentiation between 'exercises' and 'problems'. According to Polya (1973), Schoenfeld (1985), and Charles & Lester (1982) mathematical task is a problem when it incorporates a challenge for learners. It should (a) be motivating; (b) not include readily available procedures; (c) require an attempt; and (d) have several approaches to solution. "Obviously, these criteria are relative and subjective with respect to a person's problem-solving expertise in a particular field, i.e. a task that is cognitively

demanding for one person may be trivial (or vice versa) for another" (Leikin, 2004, p. 209).

Teachers' role in devolution of challenging mathematical tasks

Jaworski (1994) claimed that mathematical challenge together with sensitivity to students, and management of learning are core elements of teaching. In order to develop pupil's mathematical understanding a teacher must create situations that demand from the students' mental effort. Teachers' choices of mathematical tasks for their classes and the ways in which these tasks are introduced to students determine the quality of mathematics in the classroom (e.g., Simon, 1997; Steinbring, 1998). However, many teachers choose conventional tasks for their lessons and guide students towards 'standard' solutions (Leikin & Levav-Waynberg, accepted). Simply providing teachers with ready-to-use challenging mathematical activities is not sufficient for their implementation. The teachers should be aware and convinced of the importance of mathematical challenge in teaching and learning mathematics, they should 'feel safe' when dealing with such kind of mathematics (mathematically and pedagogically) and have autonomy (Krainer, 2001) in employing this kind of mathematics in their classes.

Teachers' knowledge and beliefs

Teachers' knowledge and beliefs are interrelated and both have very complex structure that determines teachers' decision making when planning, performing, and reflecting. In this study we consider mathematical challenges as an integral part of teachers' content knowledge (Shulman, 1986). From this point of view *teachers' subject-matter knowledge* comprises their own understanding of the essence of mathematical challenge, their knowledge of challenging mathematics, their ability to approach challenging tasks. In this study teachers' conceptions of mathematical challenge are an integral part of teacher's subject-matter knowledge attributed to their meta-analysis of mathematical content to be taught. *Teachers' pedagogical content knowledge* includes knowledge of how students cope with challenging mathematics, as well as knowledge of appropriate learning setting. *Teachers' curricular content knowledge* includes knowledge of different types of curricula and understanding different approaches to teaching challenging mathematics.

THE STUDY

The purpose and the questions

This is the first stage of an ongoing study that analyses development of teachers' conceptions of mathematical challenge. At this stage we explored teachers' views of mathematical challenges through the content analysis of their definitions and examples. The main research question here is: What kinds of mathematical challenge teachers mention in their definitions? What types of mathematical problems teacher provide as examples of challenging tasks? How teachers rank different criteria of

mathematical challenge? How teachers rank mathematical tasks with respect to the challenge they embrace?

Procedure and tools

Two groups of in-service secondary school mathematics teachers with at least 10 years of experience took part in our study. Group A included 9 teachers, Group B included 41 teachers. There were two stages in this research:

Stage 1: The teachers from Group A were asked to complete an open Questionnaire-1 that included 2 assignments: (1) to define mathematical challenge; (2) to give two examples of mathematically challenging tasks and explain how they illustrate mathematical challenge (for the effectiveness of examples as a research tool see Zazkis & Leikin, accepted). After completing Questionnaire-1 the teachers discussed their definitions and examples. They were asked to solve the tasks given as examples by other teachers and evaluate challenge incorporated in the tasks. Lastly the teachers were asked to choose (at home) 3 types of mathematical challenge and present an example of each one. All the teachers' work was collected and the discussion was recorded in hand-written protocol. Teachers' written responses and their discourse during the discussion were analysed by two researchers independently. Based on this analysis we identified 12 categories of challenging mathematical tasks as viewed by teachers (see Table 1). Based on this categorization Questionnaire-2 was constructed. We also included in Questionnaire-2 three problems that were suggested by the teachers from Group A as "challenging at different levels" and "corresponding to different characteristics of mathematical challenge".

Stage 2: Forty-one teachers from Group B were presented with Questionnaire-2 and asked (1) to rank types of the tasks with respect to the level of their challenge from 1 (the most challenging) to 12 (the least challenging); (2) to solve and rank the three chosen tasks from 1 (the most challenging) to 3 (the least challenging) with respect to teaching 9th grade students.

RESULTS

Characteristics of challenging mathematical tasks

Teachers' definitions included various criteria for mathematical challenge. For example, Roni and Sohila addressed solving problems in different ways, Soha considered non-conventional problems, Tami clearly indicated that using combination of different topics in one solution is challenging. Sohila also addressed finding mistakes in solutions of the problems as a mathematical challenge.

Roni: Mathematical challenge for me is a problem that has several stages in solution or problem that has different solutions. The solution is not obvious.

Soha: Mathematical challenge requires thinking at the level higher than regular. A challenging problem is a problem whose solution or topic is non-standard. I think that mathematical challenge is relative. It is difficult to characterize a problem as challenging or not challenging without

considering population. The problem can be challenging for the 1st grade students and not challenging for the 3rd graders. The problem may be challenging for 'regular' people but not challenging for mathematicians.

Tami: Mathematical challenge is a combination of a number of different methods and topics together. It means that solving challenging problem I use different mathematical principles and topics (e.g. algebra and geometry or analytical geometry)

Sohila: For me mathematical challenge is (a) looking for different solutions of a problem or (b) looking for the mistakes in solutions.

As mentioned earlier we identified 12 categories based on the analysis of teachers' definitions of challenging mathematical tasks (see Table 1).

Table 1: Frequency of different characteristics in Group A and ranking the characteristics by Group B.

No of the criteria in quest-2	Characteristics of mathematical challenge	Group A			Group B	
		Criteria mentioned in the definition Lesson -1	Problems of a particular type Lesson -1	Problems of a particular type HW-1	Appeared as the 1 st choice	Average rank
6.	A problem that requires combination different mathematical topics	2		3	4	6
2.	A problem that requires logical reasoning	1	2	1	11	6.15
1.	A problem that has to be solved in different ways	7	2	4	5	6.44
12.	An inquiry based problem	1	4	6	7	6.59
7.	A non-conventional problem	1	6	2	9	7.66
10.	A problem that requires generalization of problem results		1	1	3	8.20
3.	Proving a new mathematics statement			3		8.29
9.	A problem that requires auxiliary constructions			1	2	9.90
11.	Finding mistakes in solutions	1				10.15
8.	A paradox	1	2	2		10.29
5.	A problem that requires knowledge of extra-curricula topics			1		10.56
4.	A problem with parameters			2		10.78

Table 1 presents frequency of the appearance of different characteristics in the definitions and problems given by teachers in group A and ranking the criteria by Group B. Table 1 also presents distribution of the examples of challenging problems given by teachers in group A with respect to different characteristics. This characterization of the problems was performed by the teachers themselves and by the two researches independently and then was discussed and refined in group

discussion. Within the space limit of this paper we cannot consider the differences between the kinds of problems the teachers presented on the spot and at home. Note that whereas only one teacher mentioned inquiry as a characteristic of challenging problem, ten of forty-five problems suggested by the teachers were open inquiry problems (e.g., Task 3).

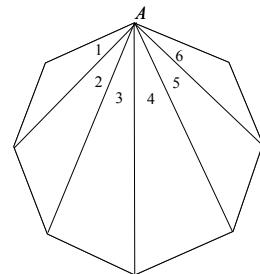
Ranking the characteristics

The teachers in group B were asked to score the criteria that arose from the analysis of the responses of teachers from group A. We found that teachers' views on mathematical challenge as followed from the teachers ranking were consistent with the views of teachers from group A. The minimal average rank was given to combination of different topics and tools in one particular solution. Logical reasoning was ranked as most suitable criteria for mathematical challenge by 11 of 41 teachers. The three other criteria that were highly ranked by teachers from group B were problem solving in different ways, mathematical inquiry and non-conventional tasks. We were surprised by the fact that non-conventional task were scored lower than other abovementioned criteria. However, from teachers' discussion it became clear that non-conventional tasks were less popular since teachers often did not feel 'safe' enough to use this type of tasks in their classes.

Examples of challenging problems

Task 1:

In a regular octagon all the diagonals from vertex A are constructed. By this construction six disjoint angles are obtained near the vertex A. What can you say about these angles:
 $\angle A_1, \angle A_2, \angle A_3, \angle A_4, \angle A_5, \angle A_6$



Task 2:

Find the product of the two numbers: $407 \cdot 393$

Task 3:

The distance between school and Tom's home is 9 km and the distance between the school and Jerry's home is 7 km. What is the distance between Tom's and Jerry's homes?

Figure 1

All the problems given by the teachers in group A (including Tasks 1, 2 and 3) were discussed by them in the whole group discussion. Before the discussion they were asked to solve the problems given by other teachers and evaluate the level of their challenge. Often, when solving, the teachers did not find problems of other teachers challenging at all. They claimed the problems were too easy. However, after the 'authors' of the problems explained why they considered the problems challenging the teachers accepted all of them as examples of challenging tasks.

As a result of the whole group discussion (in group A) the teachers developed their comprehension of the relativity of mathematical challenge. Similarly to considerations presented by Soha in her definition of mathematical challenge, teachers agreed that in order to evaluate mathematical challenge of a problem it should be considered with respect to students' age, knowledge, ability, expertise and creativity.

Tami: You cannot say whether the task is challenging if you do not know who the students are, what they know, where they lean and moreover who is the teacher that teaches them. For some students this will be a challenge, for other students not at all.

Figure 1 presents three problems that were provided by the teachers in group A and chosen for Questionnaire-2. The reasons for this choice were the following: The three tasks represented *different types of challenges* as shown above, they belong to *different fields* of school mathematical curriculum, and were found challenging for the *same grade level*: 9th grade.

Task 1 has different solutions, requires use of different topics in one solution (i.e. circle and inscribed regular polygons or calculation of angles and equilateral triangles and trapezoid, properties of an angles bisector). This problem also requires auxiliary constructions.

Task 2 is not standard task since its 'elegant' solution is based on use of reduced multiplication formulas.

Task 3 is an example of an open problem that has infinite number of solutions that should be presented in the form of inequation ("something that you do not see in the textbook"). This is an inquiry problem whose formal solution is not trivial even for the teachers (see Figure 2). Teachers agreed that this problem required logical consideration in the process of solution.

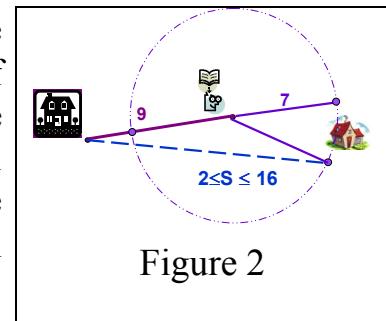


Figure 2

The teachers in group B ranked the challenge of the tasks (from 1 – the most challenging to 3 – the least challenging). Table 2 shows teachers' performance in the three tasks and their ranking. According to the teachers' ranking Task 3 appeared to be the most challenging one (1.65). The least challenging was Task 2 (2.19).

We found clear relationship between the difficulty of the tasks for the teachers and their perception of the level of their challenge. Only 3 of 41 teachers gave completely correct answer to Task 3 whereas 38 of 41 teachers solved Task 2 correctly including 34 teachers who used the 'elegant' solution.

Evaluation of the challenge of the tasks by teachers in group A correlated with teachers' views in group B: the more challenging characteristics a task comprised in the eyes of the teachers' from group A the more challenging it was ranked by the teachers from group B. Additionally, the most challenging task (task 3) fitted five

most 'popular' characteristics of challenge as ranked by teachers from group B whereas Task 2 fitted only two of these characteristics.

Table 2: Teacher's solutions and challenging rank of 3 tasks (Group B)

	Task 1		Task 2		Task 3	
	Answer	No of teachers	Answer	No of teachers	Answer	No of teachers
Correct solution	22.5°	20	$ \begin{aligned} 407 \cdot 393 &= \\ &= (400+7)(400-7) = \\ &= 160000-49 = 159951 \\ &\text{159951 by long multiplication} \end{aligned} $	34 4	$2 \leq x \leq 16$	3
Partly correct solution	22.5° - no justification	7		0	$2 < x < 16$	11
	$\angle 1 = \angle 6, \angle 2 = \angle 5, \angle 3 = \angle 4$	8			There are infinite number of situation	12
Incorrect solution	No answer	6	No answer	3	2, 16, $\sqrt{130}$ or There are number of cases	7
					2 or 16	2
					No answer	6
					Challenge rank (from 1 the highest to 3 the lowest):	
	2.06		2.19		1.65	

Concluding remarks

The experiment presented in this paper is only the first stage in the study on teachers' conceptions of challenging mathematics. Our study demonstrates that teachers (as a group) held a broad conception of mathematical challenge. During the group discussion they refine and verify their conceptions and attain a shared meaning. We were happy to see that many of the criteria teachers suggested for mathematical challenge had been used by Krutetskii in his study on students' mathematical abilities (Krutetskii, 1976).

We found that teachers connect pedagogy and mathematics (Jaworski, 1992; Shulman, 1986) incorporated in the mathematical tasks: when reasoning about mathematical challenge the teachers were sensitive to individual differences among their students. This connection was also evident in their reasoning about the relativity of the challenge. During the discussions the teachers clearly stated that they value mathematical challenge and agreed that challenging mathematics is important for 'deepening and broadening of students' mathematical thinking'.

At the time of this study the teachers from Group A participated in MA program for mathematics teachers. We were glad to see that they used the content of the courses in their reasoning about challenges, since most of the courses in the program were challenging to them. For example, the frequency of the requirement for different solutions (mentioned by 7 of 9 teachers) may be explained by the fact that teachers in

group A were aware of the special interest of the researchers in different ways of problem solving (e.g., Leikin & Levav-Waynberg, accepted). Additionally, one of the tasks presented by the teachers as a challenging task was similar to the one used in the studies by Verschaffel & De Corte (e.g., Verschaffel & De Corte, 1997). This implementation of knowledge obtained in a different context indicates for us the importance of the systematic teachers education.

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SEMIOTIC GAMES: THE ROLE OF THE TEACHER

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The paper uses a semiotic lens to interpret the interactions between teacher and students, who work in small collaborative groups. This approach allows focussing some important strategies, called semiotic games, used by the teacher to support students mathematics learning. The semiotic games are discussed within a Vygotskyan frame.

INTRODUCTION

The role of the teacher in promoting learning processes is crucial and has been analysed according to different frameworks. For example the Theory of Didactic Situations, originated by G. Brousseau (1997), defines the teacher as a *didactical engineer*. (S)he designs the situations and organises the *milieu* according to the piece of mathematics to be taught and to the features of the students (*mesogenesis*: see Sensevy et al., 2005); (s)he divides the activity between the teacher and the students, according to their potentialities (*topogenesis*); moreover the classroom interactions are pictured according to the *didactic contract*, that is the system of reciprocal explicit and implicit expectations between the teacher and the students as regards mathematical knowledge. Brown and McIntyre (1993) underline that the teacher works with students in classrooms and designs activity for classrooms using her/his *craft knowledge*, namely a knowledge largely rooted in the practice of teaching. Siemon & al. (2004, p. 193) point out “the need for a deeper understanding of the ways in which teachers contribute to the shaping of classroom cultures”. Other researchers, who work according to Vygotsky’s conceptualization of ZPD (Vygotsky, 1978, p. 84), underline that teaching consists in a process of enabling students’ *potential* achievements. The teacher must provide the suitable pedagogical mediation for students’ appropriation of *scientific concepts* (Schmittau, 2003). Within such an approach, some researchers (e.g. Bartolini & Mariotti, to appear) picture the teacher as a *semiotic mediator*, who promotes the evolution of *signs* in the classroom from the personal senses that the students give to them towards the scientific shared sense. In this case teaching is generally conceived as a system of actions that promote suitable processes of *internalisation*.

Our approach is in the Vygotskian stream: the teacher is seen as a semiotic mediator, who promotes students’ internalisation processes through signs. But some changes are proposed with respect to the classical Vygotskyan approach. First, we extend the notion of sign to all *semiotic resources* used in the teaching activities: words (in oral or in written form); extra-linguistic modes of expression (gestures, glances, ...); different types of inscriptions (drawings, sketches, graphs, ...); different instruments (from the pencil to the most sophisticated ICT devices). Second, we consider the *embodied* and *multimodal* ways in which such resources are produced, developed and

used. Within such a framework, we utilise a wider semiotic lens (the *semiotic bundle*, sketched below) to focus the interactions between teacher and students. Our semiotic lens allows framing and describing an important semiotic phenomenon, which we call *semiotic games*. The semiotic games practise is rooted in the craft knowledge of the teacher, and most of times is pursued unconsciously by her/him. Once explicit, it can be used to properly design the teacher's intervention strategies in the classroom for supporting students' internalisation processes.

In the following three sections we discuss: (i) the multimodal paradigm and the semiotic tools suitable for describing mathematics learning processes; (ii) an emblematic example, through which the notion of semiotic games is introduced (the main result of the paper); (iii) some didactical consequences.

FROM THE MULTIMODALITY OF LEARNING PROCESSES TO THE SEMIOTIC BUNDLE

The notion of *multimodality* has evolved within the paradigm of *embodiment*, which has been developed in these last years (Wilson, 2002). Embodiment is a movement afoot in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested in how the body is involved in thinking and learning. The new stance emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment. A major consequence is that the boundaries among perception, action and cognition become *porous* (Seitz, 2000). Concepts are so analysed not on the basis of "formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world" (Gallese & Lakoff, 2005, p. 455), but considering the *multimodality* of our cognitive performances. Verbal language itself (e.g. metaphorical productions) is part of these cognitive multimodal activities (*ibid.*).

Semiotics is a powerful tool for observing the didactical processes. However, the classical semiotic approaches put strong limitations upon the structure of the semiotic systems they consider. These result too narrow for interpreting the didactical phenomena in the classroom. This happens for two reasons:

- (i) Students and teachers use a variety of semiotic resources in the classroom: speech, gestures, glances, inscriptions and extra-linguistic modes of expression. But some of them do not satisfy the requirements of the classical definitions for semiotic systems as discussed in the literature (e.g. in Duval, 2006 or in Arzarello, 2006).
- (ii) The way in which such different resources are activated is multimodal. It is necessary to carefully study the relationships within and among those, which are active at the same moment, and their dynamic developing in time.

Hence we need a broader theoretical tool for analysing the semiotic resources in the classroom. This tool is the *Semiotic Bundle*, introduced in Arzarello (2006). It encompasses all the classical semiotic systems as particular cases. Hence, it is coherent with the classical semiotic analysis, but broaden it and allows getting new

results and framing the old ones within a unitary wider picture. Roughly speaking (for a full description see Arzarello, 2006), while the classical semiotic systems concern very structured systems, whose rules of sign production and manipulation are very precise algorithms (from the oral and written language to the algebraic or Cartesian register) the semiotic bundle includes all signs produced by actions that have an intentional character (e.g. speaking, writing, drawing, gesticulating, handling an artefact, etc.) and whose modes of production and transformation (e.g. for gesturing or drawing) may encompass also approaches less deterministic and more idiosyncratic than algorithms. A semiotic bundle is a dynamic structure, where such different resources coexist and develop with their mutual relationships, according to the multimodal paradigm. Hence it allows considering a variety of resources, which span from the compositional systems, usually studied in traditional semiotics (e.g. formal languages) to the open sets of signs (e.g. sketches, drawings, gestures). An example of semiotic bundle is represented by the unity speech-gesture. It has been written that “gesture and language are one system” (McNeill, 1992, p.2): from our point of view, gesture and language are two components of the same semiotic bundle. Research on gestures has already shown important relationships between them (e.g. *match* Vs. *mismatch*, see Goldin-Meadow, 2003).

We have used the semiotic bundle to analyse different classroom stories (Arzarello et al., 2006; Arzarello, 2006). It has revealed particularly useful for studying several didactic phenomena that happen in the classroom, especially some interactions between teacher and students, who work in small groups. We have called them the *semiotic games*. They consist in strategies of intervention in the classroom that many times the teacher activates unconsciously; once (s)he becomes aware of such strategies, s(he) can use them in a more scientific way for improving her/his students achievements. We shall introduce the semiotic games through an emblematic example in the following section; further discussion is found in Sabena (2007) and Arzarello & Robutti (to appear). We have observed such games in different classes and with students of different ages (from elementary to secondary school).

THE SEMIOTIC GAMES THROUGH AN EMBLEMATIC EXAMPLE

The activity we shall comment concerns students attending the third year of secondary school (11th grade; 16-17 years old). They attend a scientific course with 5 classes of mathematics per week, including the use of computers with mathematical software. These students are early introduced to the fundamental concepts of *Calculus* since the beginning of high school (9th grade); they have the habit of using different types of software (Excel, Derive, Cabri_Géomètre, TI-Interactive, Graphic Calculus: see Arzarello et al. 2006) to represent functions, both using their Cartesian graphs and their algebraic representations. Students are familiar with problem solving activities, as well as with interactions in small groups. The methodology of mathematical discussion is aimed at favouring the social interaction and the construction of a shared knowledge.

We will comment some excerpts from the activity of a group of three students: C, G, S. They are clever pupils, who participate to classroom activities with interest and active involvement. In the episodes we present, there is also the teacher (T), whose role is crucial and will be carefully analysed: he is not always with these students, but passes from one group to the other (the class has been divided into 6 small groups of 3-4 students each). The excerpts illustrate what is happening after the group has done some exploring activities on one PC, where Graphic Calculus produces the graphs of Figure 1. Their task is to explain the reasons why the slope of the ‘quasi-tangent’ (see the box with Fig. 1) is changing in that way. The students know the concepts of increasing/decreasing functions but they do not yet know the formal notion of derivatives. Moreover they are able in using Graphic Calculus and know that the ‘quasi-tangent’ is not the real tangent, because of discrete approximations.

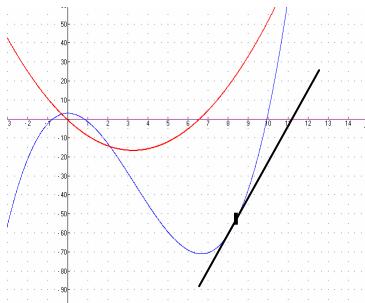


Figure 1

The two graphs represent: the function $f: x \rightarrow 0.5x^3 - 5x^2 + 3$; the slope of the ‘quasi-tangent’ to the function f [i.e. the secant through a moving point $(x, f(x))$ and the point $((x+\delta, f(x+\delta))$, for very small δ]. Graphic Calculus allows to see the genesis of the two graphs through a point P, which moves and traces the graph of the function f . In the same time one can see the corresponding moving ‘quasi-tangent’ and the graph of its slope, which is traced in real time while P is moving.

Typically their first explanations are confuse (see Episode A) and expressed in a semiotic bundle, where the speech is not the fundamental part. In fact, the main component of the semiotic bundle consists in the multimodal use of different resources, especially gestures, to figure out what happens on the screen. Figures 2 show how C captures and embodies the inscriptions in the screen through his gestures. More precisely, the evolution of the gesture from Fig. 2a to Fig. 2e illustrates a sprouting concept not completely formulated in words. It could be phrased so: “*the quasi-tangent is joining pairs of points whose x-coordinates are equidistant, but it is not the same for the corresponding y-coordinates: the farther they are the steepest is the quasi-tangent*”. These ideas are jointly expressed through gestures and words. In fact, C’s words refer only to the ‘quasi-tangent’ line and express the fact that the interval Δx is always the same; the remaining part is expressed through gestures. Only later the concept will be expressed verbally. We call the gesture in Fig. 2b the *basic sign* (the thumb and the index getting near each other): in fact it triggers a semiotic genesis of signs within the semiotic bundle that is being shared among the students (and the teacher, as we shall see).

In Episode A the student C is interacting with the teacher, whose role will be analysed later. At the moment we limit ourselves to focus on the specific content of his interventions in this episode. In #1 he is echoing C’s words (#0) using a more

technical word (*delta-x*), namely he gives the scientific name to the concept expressed by C and C shows that he understands what the teacher is saying (#2). C's attention is concentrated on the relationships between the Δx and the corresponding Δy variations. Gesture and speech both contribute to express the covariation between Δx and Δy , underlining the case when the variations of Δy become bigger corresponding to fixed values of Δx . Figures 2 and the corresponding speech illustrate the multimodality of C's actions: the student is speaking and simultaneously gesturing. C grasps the relationship of covariance with some difficulty, as the misunderstanding in sentences from #5 to #9 shows. Here the intervention of the Teacher (#6) supports C in the stream of his reasoning, which can continue and culminates in sentence #11, where the gesture (see the overturned arm in Fig. 2e) gives evidence that C realised that the covariance concerns also negative slopes (but he does not use such words).

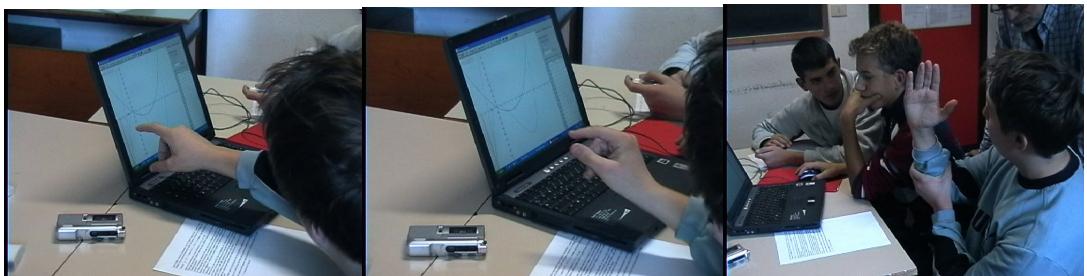


Fig. 2a (sentence #2)

Fig. 2b (sent. #2)

Fig. 2c (sent. #3)



Fig. 2d (sent. #5)

Fig. 2e (sent. #11)

Fig. 2f (sent. #21)

Episode A. (duration: 9 seconds; about one hour after the beginning of the activity).

- 0 C: The X-interval is the same ...
- 1 T: The X-interval is the same; Δx is fixed.
- 2 C: Delta...eh, indeed, however...however there are some points where... to explain it ... one can say that this straight line must join two points on the Y axis, which are farther each other. (Figs. 2a, 2b)
- 3 C: Hence it is steeper towards...(Fig. 2c)
- 4 G: Yes!
- 5 C: Let us say towards this side. When, here, ...when ...however it must join two points, which are farther, that is there is less...less distance. (Fig. 2d)
- 6 T: More or less far?
- 7 C: Less...less far [he corrects what he said in #5].
- 8 T: Ehh?

9 C: On the Y axis I am saying!
10 T: Yes!
11 C: It slants softly from this side (Fig. 2e).

After a few seconds there is an important interaction among the teacher and the students (Episode B), which is emblematic of the strategy used by the teacher to work with students for promoting and facilitating their mathematics learning.

Episode B. (duration: 34 seconds; a few seconds after Episode A).

18 T: Hence let us say, in this moment if I understood properly, with a fixed delta-x, which is a constant, ... (Fig. 2f).
19 C: Yes!
20 S: Yes!
21 T: It... is joining some points with delta-y, which are near (Fig. 2f).
22 C: In fact, now they [the points on the graph] are more and more...
23 T: It is decreasing, is it so? [with reference to Δy]
24 S: Yes!
25 C: ...they [their ordinates] are less and less far. In fact, the slope... I do not know how to say it,.....the slope is going towards zero degrees.
26 T: Uh, uh.
27 C: Let us say so...
28 S: Ok, at a certain point here delta-y over delta-x reaches here...
29 C: ...the points are less and less far.
30 T: Sure!
31 S: ...a point, which is zero.

[sentences of C(#27, #29), of S(#28, #31) and of T (#30) are intertwined each other]

This episode shows an important aspect of the teacher's role: his interventions are crucial to foster the positive development of the situation. This appears both in his gestures and in his speech. In fact he summarises the fundamental facts that the students have already pointed out: the covariance between Δx and Δy and the trend of this relationship nearby the stationary point (we skipped this part). To do so, he exploits the expressive power of the semiotic bundle used by C and S. In fact he uses twice the basic sign: in #18 to underline the fixed Δx and in #21 to refer to the corresponding Δy and to its smallness nearby the local maximum x (non redundant gestures: see Kita, 2000). In the second part of the episode (from #22 on) we see the immediate consequence of the strategy used by the teacher. C has understood the relationship between the covariance and the phenomena seen on the screen nearby the stationary point. But once more he is (#25) unable to express the concept through speech. On the contrary, S uses the words previously introduced by the teacher (#18, #21) and converts what C was expressing in a multimodal way through gestures and (metaphoric) speech into a fresh semiotic register. His words in fact are an oral form of the symbolic language of mathematics: the semiotic bundle now contains the official language of Calculus. His sentences #28, #31 represent this formula. The episode illustrates what we call semiotic games of the teacher. Typically, the teacher uses the multimodality of the semiotic bundle produced by the students to develop his

semiotic mediation. Let us consider #18 and #21 and Fig. 2f. The teacher mimics one of the signs produced in that moment by the students (the basic sign) but simultaneously he uses different words: precisely, while the students use an imprecise verbal explanation of the mathematical situation, he introduces precise words to describe it (#18, #21, #23) or to confirm the words of S (#30). Namely, the teacher uses one of the shared resources (gestures) to enter in a consonant communicative attitude with his students and another one (speech) to push them towards the scientific meaning of what they are considering. This strategy is developed when the non verbal resources utilised by the students reveal to the teacher that they are in ZPD. Typically, the students explain a new mathematical situation producing simultaneously gestures and speech (or other signs) within a semiotic bundle: their explanation through gestures seems promising but their words are very imprecise or wrong and the teacher mimics the former but pushes the latter towards the right form.

CONCLUSIONS

Semiotic games are typical communication strategies among subjects, who share the same semiotic resources in a specific situation. The teacher uses the semiotic bundle both as a tool to diagnose the ZPD of his students and as a shared store of semiotic resources. Through them he can develop his *semiotic mediation*, which pushes their knowledge towards the scientific one. Roughly speaking, semiotic games seem good for focussing further how “the signs act as an instrument of psychological activity in a manner analogous to the role of a tool in labour” (Vygotsky, 1978, p. 52) and how the teacher can promote their production and internalisation. The space does not allow to give more details (they are in Arzarello & Robutti, to appear) and we limit ourselves to sketchily draw some didactical consequences. A first point is that students are exposed in classrooms to cultural and institutional signs that they do not control so much. A second point is that learning consists in students’ personal appropriation of the signs meaning, fostered by strong social interactions, under the coaching of the teacher. As a consequence, their gestures within the semiotic bundle (included their relationships with the other signs alive in the bundle) become a powerful mediating tool between signs and thought. From a functional point of view, gestures can act as “personal signs”; while the semiotic game of the teacher starts from them to support the transition to their scientific meaning. Semiotic games constitute an important step in the process of appropriation of the culturally shared meaning of signs, that is they are an important step in learning. They give the students the opportunity of entering in resonance with teacher’s language and through it with the institutional knowledge. However, in order that such opportunities can be concretely accomplished, the teacher must be aware of the role that multimodality and semiotic games can play in communicating and in productive thinking. Awareness is necessary for reproducing the conditions that foster positive didactic experiences and for adapting the intervention techniques to the specific didactic activity. E.g. in this report we have considered teacher’s interventions in small collaborative groups. In a whole class discussion, the typology of semiotic games to

develop is likely to change, depending on the relationships within and among the different components of the semiotic bundles produced and shared in the classroom. This issue suggests new researches on the role of the teacher in the classroom, where the semiotic lens can once more constitute a crucial investigating tool.

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EXAMPLES, A MISSING LINK

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The purpose of this paper is to draw attention to a missing link regarding the problems involved in ‘generating’ an example of a defined concept. Through examining students’ conceptions while generating (an example), it is argued that these conceptions might be separately linked to a hidden activity: checking (the status of something for being an example). Finally, the missing link between students’ conceptions while generating and students’ conceptions while checking will be discussed.

INTRODUCTION

Using examples is so blended with our standard practice of teaching mathematics that what is written about the importance of using examples seems to be an expression of triviality. However, to the very same extent that making use of examples seems trivial and mundane, choosing a suitable collection of examples is problematic. It seems that any choice of examples bears an inherent *asymmetric* aspect, i.e. while for the teacher they are examples of certain relevant generalisations transferable to other examples to be met, for the students they could remain irrelevant to the target generalization (Mason and Pimm, 1984). When the intended generalization is a concept, usually accompanied by a definition, this divorce of examples from what they exemplify is mainly shown in the literature by examining how students tackle *checking* problems, i.e. checking the status of something for being an example (Tall and Vinner, 1981).

Contrary to the widespread standard teaching practice in which new concepts are introduced by and through teacher-prepared examples accompanied by his or her commentaries on what is worth considering in the prepared examples, there are a few and still experimental non-standard settings in which students are encouraged from the outset to *generate* their own examples. These works mainly arise from the perspective that “ mathematics is a constructive activity and is most richly learnt when learners are actively constructing objects, relations, questions, problems and meanings” (Watson and Mason, 2005, p. ix).

The previous two distinct paragraphs, one concerning *checking* and the other concerning *generating*, metaphorically stand for the current view of the literature on these two processes, mainly as two distinct processes (for the examples of this separation see Dahlberg and Housman, 1997, or, Hazzan and Zazkis, 1997). Calling into question this widespread separation between checking and generating is the main theme of the present paper. It will be argued that as far as generating is concerned, its separation from checking lies in the learner’s *conception* (of the underlying concepts) and the learner’s generating approach rather than the designer’s (researcher or teacher) will.

BACKGROUND

This paper is based on a wider study aimed at investigating students' understanding of equivalence relations (Asghari and Tall, 2005). The following task (The Mad Dictator Task) was originally designed while having the standard definition of equivalence relations in mind. The task was tried out on twenty students with varied background experience, none of them had any *formal* previous experience of equivalence relations and related concepts. In a one-to-one interview situation each student was introduced to the definition of a 'visiting law' (see below). Then each student was asked to give an example of a visiting law on the prepared grids (see below).

The Mad Dictator Task

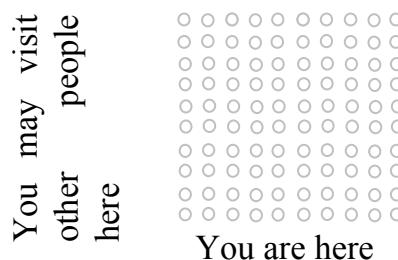
A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.

A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.

A visiting law must obey two conditions to satisfy the mad dictator:

1. When you are in a particular city, you are allowed to visit other people in that city.
2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.



In the previous paper mentioned above, most of the data came from the students' involvement in generating an example. However, in that paper all the tasks involved, including generating, were in the background while students' conceptions of the concepts of interest were in the fore. Now, in this paper, I turn my attention around and start scrutinizing the tasks.

As far as generating and checking are concerned there are a few methodological points worthy of consideration:

First, in our earlier paper mentioned above, according to a methodological choice, the focus of the study was on the outcomes of learning (learned) rather than on the learners. In the present paper, I give more weight to the individuals' voices. However, again it is not the individuals per se that matter; the focus will be on what they reveal about the possible interconnections between generating, checking and the students' conceptions.

Second, in this study generating an example basically means coming up with certain points on the grid where there are incredible blind choices (two to the power of hundred different ways of putting the points on the grid), and where only a tiny portion of all the possible choices constitutes the potential example space (something about one over two to the power of eighty). However, the number of objects (examples) in this tiny portion is still big enough (it is exactly 115975) to surprise the participants by the potential possibility of having an example that is open to them.

Third, in the course of the interview, after generating the first examples each interviewee was asked to generate another one. However, the number of generated examples was mainly determined by the interviewee's will rather than any predetermined plan. As a result, the interviewees' works range from generating only two examples to suggesting a way to generate an example, though they were never asked to explain a general way to generate an example.

Fourth, one of the most subtle points of each interview was to make a decision about checking point, i.e. whether the interviewer should ask the interviewee to check whether his or her self-generated figure is an example or not. This decision was entirely contingent on the interviewee's way of generating his or her example. As a matter of fact, to the same extent that the interviewer could not be aware of all possible conceptions in advance, he could not examine all possible contingencies beforehand. As a general rule, if the interviewer sensed that the way of generating an example reflects a new conception (at least new from the interviewer's point of view) he asked for checking, otherwise that decision was left to the interviewee.

Indeed, the fourth point underlines the missing link stressing in this paper. For a long time, I was not aware of the hidden existence of checking while generating. As a result, generating an example came to a halt as soon as student's generated figure seemed to be an example. However, students' spontaneous attempts to justify their generated figure brought to the fore some complex interrelationships between generating and checking. The next section exemplifies some of these complexities.

Exemplification and Conception

As mentioned before, twenty students with varied background experiences participated in this study. However, in the present paper we only follow two students (Dick and Hess) across two different tasks, namely generating and checking. This is not because these two students and/or their work are typical. They just exemplify how a certain conception may carry different weights in different tasks.

I shall start with Dick (at the time, a first year undergraduate law student) when he was *generating* his first example.

Dick: this question's been specially designed to confuse.

After a few minutes puzzling over the task and putting some points on the grid, he realized how to represent the first condition:

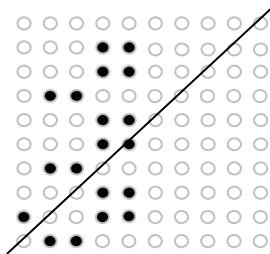
Dick: that line (diagonal), that line is the first condition, because you allow to visit people in that city, you see.

Now, his grid looks like the figure on the right:

Dick: so let's *check* this works...um, I think you can't

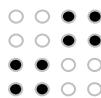
do very complicated *system*, because otherwise a *pair* of cities certainly contradict with each other,

I am not sure it works or not, I think I put that wrong, um.

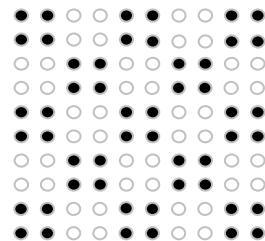


While generating and checking, Dick's focus is on a pair of cities. In other words, his work manifests a *matching conception* in which the focus is on a pair of elements.

Having failed in his first attempt, he turned his attention to a part of the grid, generated the following figure on the left, checked it matching-wise and extended it to the following figure on the right:



Dick: so *if you do a sort of cross*,
I think that satisfy the second condition,
um, I also think that would happen, because
you get line where you've already filled the
centre diagonal which I did before, which
means it satisfies the first condition.



Obviously, his figure satisfies the first condition, but, what about the second condition? Consider that the processes that he made use of was not *predictive*, i.e. making use of it did not guarantee that the product would be an example. As a result, *checking* the status of the product is inevitable. For an informed person, Dick's first example is indeed an example because it consists of two disjoint groups of mutually related elements (here, cities). However, it is not something that Dick *sees* in his figure. Yet, confined himself to matching, He checked it by taking a *random pair*:

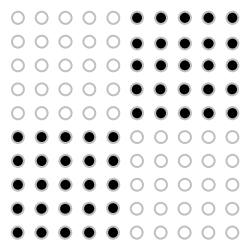
Dick: yah, so that's what I did, so every one of these dots means you can visit that city, ... and if you take any *random pair*, I go two and seven, none of them can visit same cities, um, ..., two can visit one, but seven can't visit one or two, so that's different, I think that satisfies all conditions.

But, his matching conception hindered him to generate another example (at the interviewer's request).

Dick: um, well, you have to have a *regular pattern* to obey the second condition; because otherwise, um, this is regular, if you, if I want to suppose, I can't hit how it worked out, um, ... for every pair, *there's so many pairs*, um, I can't work out, there's so many pairs that we aware if to work out another system, and I can't think another way to do it, um, I stick to this assumption, it has to be a regular system, because otherwise at least one of the pairs conflict, um...I can't think another way...it is the only example I can think of.

On the interviewer's insistence, he tries again. Yet, within a matching conception, his focus is mainly on the medium (here, the grid) in which he is about to present another example.

○ ○ ○ ○ ○ • ○ ○ • Dick: I just try a *pair*, four and seven having non
 ○ ○ ○ ○ ○ • • ○ similar, um; now, I do five and six ...
 ○ ○ ○ ○ ○ • • ○ they have to cross over with the line in the middle,
 ○ ○ ○ ○ ○ • ○ ○ ○ so I do seven, and that should...go upwards with six...
 ○ ○ ○ ○ ○ • ○ ○ ○ I think this might also work as a system.



Again, Dick needs to *check* if his system works. This time, for a moment he exhibits a different conception, i.e. *grouping*.

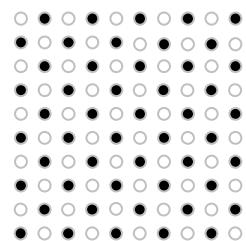
Dick: um, I just tried a *random example* and it works (ha, ha), um... I just do an example of having the first five cities all be able to visit each other, then the second five cities all be able to visit each other... once again it's a regular system.

However, Dick's grouping experience is not reflected in his next generating attempt.

Dick: um, once again I *am stuck* for other way to do it, when I did that one (first example); I thought there won't be other ways to do it, now I think of another one which works...for this one (the previous example), I worked from the top to the bottom..., now I am considering filling from left to right to the middle spots...

Soon afterwards he realised that “that would be resulted in the same sort as the second example”! But, still focusing on the figural features, he continued as follows:

Dick: if you do a system where you fill in blocks of dashes...
 something possibly diagonal lines, but *I am not sure that system work*,
 I *must fill in to see it works*...so that each circle, each visiting-city is
 not next to another visiting-city, um, so that, like that (laughing), ...
take a random pair, say three and six, not same, not same, not same...
 I think this does work actually (laughing), I just took it and it worked!



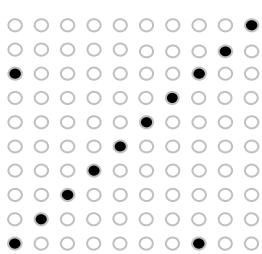
Again it is a vague sense of “a regular system” that leads Dick to his third example. As a result, *checking is inevitable*. When checking, it is only a matching procedure that connects his three examples together. When generating, not only his conception (matching) is hardly noticeable while generating his last example, but also it somehow hindered him to make it!

Dick: it seems that I shouldn't thought about it (the third example) originally because the second rule is that they must have identical or they mustn't have any in common.

Consider the dual role of the matching conception in generating and checking. It is also worth stressing that it is Dick's approach, rather than matching conception *per se*, which makes checking inevitable. Hess' experience shows this point.

Hess (at the time, a middle school student) manifested multiple ways of experiencing the situation. After generating the diagonal as his first example, he thought that “it is impossible to have another example” because if “we add another point, they would have a different point, so they never would be identical”. Soon afterwards, the same conception (matching)—once hindered him to generate an example—helped him to generate a collection of examples:

Hess: Now, if for every, for example, this point, one, if I put everywhere, for example, I put eight, I must put for eight, one; eight and one are completely the same and others are completely different...so we have infinite cases; no, it is not infinite, but it is a lot...

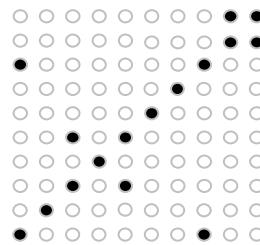


for 3, I put 5 and for 5 I put 3.

Interviewer: and still have you got the previous points?

Hess: it makes no difference...

10, I put 9, also 9, I put 10.



Now, he is looking for something more *general*, certain “property that all of them have”:

Hess: In general, any two *symmetric* points that we choose we have one (example) about this (the diagonal).

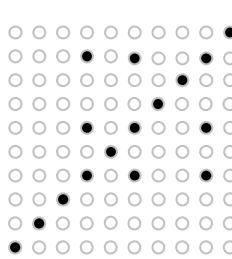
Unlike Dick, Hess envisages what an example might look like. However, like Dick, he only matches together two elements (here, two points), ignoring the possible connections that each one of these two elements might have with the other elements. As a result, some non-examples are counted as examples. However, *in action*, he chose his new pairs somehow keeping them distinct from the old ones. In action or envisaged, in Hess’ generating approach, *checking is embedded in from the outset*, yet within a matching conception.

Later on, Hess generated a non-example symmetric figure. To do so, he needed to give weight to an element, and then, its likely relations with more than one element.

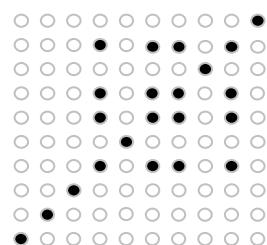
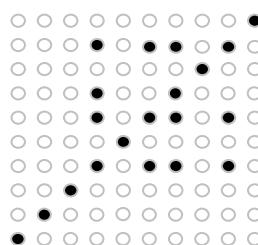
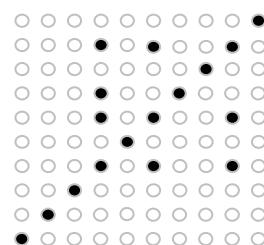
Hess: if 10 has something, that one has 10 too; If 10 has 4, 4 has 10 too, then I must prove that 4 and 10 are equal...if 10 has 5, no, it was rejected... 10 and 4 no longer are equal.

His attempt to provide a non-example symmetric figure plants the seeds of a group-conception reflected in his next generating attempt. Let us enter his work from the middle where he expressed what he has got “as clear as possible and very nicely”.

He has already generated the following example on the left:



Hess: My hypothesis is that this one has the conditions of the problem, well, it has the conditions, I suppose I want to make a new condition (a new example) from the previous condition; suppose I choose another point somewhere, I add 4 and 7. (He continues) now, *seven is a member of four*. Then I am working on the seventh column, because seven had itself I determine all the other points but seven; (those are) nine, six and four. (He continues) then in each of the others, I determine seven, because it has just been added.



Hess sees no point to check his newly generated figure. He even does not see his example separated from the process leading to it.

Interviewer: How you know that this figure is an example?

Hess: I don't want to show it is an example, we suppose it is an example, then I add, and I prove when it is added (the conditions) are satisfied again.

Interviewer: the question is here, you want to give it to the dictator, and say it is the way that people can visit each other; the dictator look at it and say whether you have satisfied the laws. He doesn't want to change it; he saves your paper. You are a dictator; do you accept this one without any change?

Hess' generating attempt manifested a group of pair-wise related elements. To check his figure (on the interviewer's insistence), he turns back to matching for a moment before experiencing a group with a *focal* element related to all the other elements of the group:

Hess: each two we consider, either they are alike or they are different...Yes. I prove it like this...we investigate for each column, those that are equal to it, those that must be equal to it, are they equal to it, or not?

As it can be seen, one conception is reflected when generating and another conception when checking. Reconciling these conceptions means reconciling generating and checking.

Conclusion

The fragmented experiences described in this paper suggest something in line with what Marton and Booth call *the path of learning*: "that learning proceeds from a vague undifferentiated whole to a differentiated and integrated structure of ordered parts...the more that this principle applies in the individual case, the more successful is the learning that occurs" (Marton and Booth, 1997, p.138).

However, it does not mean that the aspects that have been differentiated and integrated when handling a certain task are also being carried to another task. They are *usage-specific*. In the case of our interest, this means there are certain interconnections between *what* a student conceptualizes and *how* he or she generates an example. In the same vein, there are certain interconnections between *what* he or she conceptualizes and *how* he or she checks the status of something for being an example. This suggests the following schematic figure:



What is experience on the left figure is not necessarily the same as what is experienced on the right figure. They vary in certain aspects; altogether manifest the variation in the students' experiences of the concept involved. Reconciling different aspects of this diversity may result in reconciling generating and checking, and consequently there would be no need for checking after generating. However, this does not mean that there

would be no need for checking activities. Indeed, we need them alongside generating activities, since it is only in the course of tackling different tasks that different aspects of a concept may be differentiated and integrated. It is the way that students may find the missing links between generating and checking.

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SCAFFOLDING REVISITED: FROM TOOL FOR RESULT TO TOOL-AND-RESULT

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The metaphor of 'scaffolding' is popular in mathematics education, particularly in accounts purporting to examine the mediated nature of learning. Drawing on Holtzman and Newman's interpretation of Vygotsky I argue that scaffolding rests on a dualistic view that separates the knower from the known. In line with their work, I begin to explore what an alternative metaphor – development through performance (in the theatrical sense) might mean for mathematics education .

INTRODUCTION

Years ago, my colleagues, Joan Bliss and Sheila Macrae, and I conducted a study into the notion of 'scaffolding' (Bruner 1985) in primary (elementary) school mathematics and science. At the time we had difficulty finding any evidence of scaffolding in practice! What we observed in lessons could just as easily be categorized as 'explaining' or 'showing'. Scaffolding, in the sense of providing a support for the learner, a support that could be removed to leave the structure of learning to stand on its own, was elusive (Bliss, Askew & Macrae, 1996).

At the time this elusiveness seemed to be a result of two things. Firstly the nature of what was to be learnt. Many of the examples in relevant literature (for example, Rogoff (1990), Lave and Wenger (1991)) attend to learning that has clear, concrete outcomes. Becoming tailors or weavers means that the objects of the practice – a jacket or a basket – are apparent to the learner: the apprentice 'knows' in advance of being able to do it, what it is that is being produced. In contrast, most (if not all) of mathematics is not 'known' until the learning is done. Young children have no understanding of, say, multiplication, in advance of coming to learn about multiplication: the 'object' of the practice only become apparent after the learning has taken place. (This is not to suggest that learners do not have informal knowledge that might form the basis of an understanding of multiplication, only that such informal knowledge is different and distinct from formal knowledge of multiplication.)

Secondly, examples of scaffolding that did seem convincing focused on schooling situations that are close to apprenticeship models (Lave and Wenger, *ibid.*) in that the teacher-learner interaction is mainly one-to-one. For example, Clay (1990) provides a strong Vygotskian account of "Reading Recovery": a programme based on individual instruction.

This led me to turn away from the Vygotskian perspectives – all very nice in theory, but did they have much to offer the harried teacher of 30 or more students? Recently I've returned to consider aspects of the work of Vygotsky, particularly as interpreted

by the work of Lois Holzman and Fred Newman. They argue, amongst other things, to attend more to Vygotsky's ideas of tool-and-result, and the argument that this leads to a 'performatory' (in the theatrical sense) approach to development.

TOOL FOR RESULT OR TOOL-AND-RESULT?

Central to the argument is Vygotsky's observation of the paradox at the heart of the psychology, in that psychology creates the very objects that it investigates.

The search for method becomes one of the most important paradoxes of the entire enterprise of understanding the uniquely human forms of psychological activity. In this case, the method is simultaneously prerequisite and product, the tool and the result of the study (Vygotsky, 1978, p. 65)

Vygotsky thus challenges the view that the method of inquiry in psychology is separate from the results of that inquiry, the traditional 'tool and result' position (Newman and Holzman, 1993). Instead

As "simultaneously tool-and-result", method is practiced, not applied. Knowledge is not separate from the activity of *practicing method*; it is not "out there" waiting to be discovered through the use of an already made tool. ... Practicing method creates the object of knowledge simultaneously with creating the tool by which that knowledge might be known. Tool-and-result come into existence together; their relationship is one of *dialectical unity*, rather than instrumental duality. (Holtzman, 1997, p. 52, original emphasis)

While Holtzman and Newman, following on from Vygotsky, challenge the view of psychology as a science akin to physical science, the same challenge can be applied to mathematics education. Models of teaching and learning based on 'mediating means' (Cole 1996) or scaffolding are largely predicated on a 'tool for result' perspective rather than a 'tool-and-result' one.

Let me illustrate this with an example. In workshops I often present teachers with Figure 1 and ask them what fraction it represents. Most say $2/5$, a few $3/5$ and fewer still suggest that it can be either. I then ask them to discuss in small groups how $1\frac{2}{3}$, $1\frac{1}{2}$, $2\frac{2}{3}$, or $2\frac{1}{2}$ can all be acceptable answers to the question 'what fraction'.

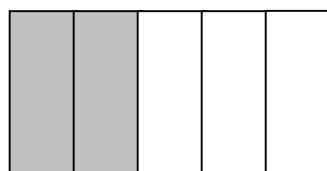


Figure 1

Initially there are puzzled looks and silence. Talk starts and gradually there are murmurs (or even cries!) of 'oh, I get it', or 'now I see it'. Back as a whole group, various 'seeings' are offered, metaphors provided ('Suppose the shaded is the amount of chocolate I have, and the unshaded the amount that you have, how many times bigger is your piece'), diagrams jotted down, and collectively we arrive at the point where most participants agree that they can 'see' the different fractions.

Looked at through the analytical lens of mediated (scaffolded) learning, and the mediation triangle (Figure 2, after Cole, *ibid.*) the ‘subject’ is the individual teacher and the ‘object’ the various ‘readings’ of the diagram. The mediated means are the talk, the metaphors, the ‘jottings’ that the teachers make in re-presenting the image for themselves. Implicit in such an account is a separation of the means and end, of the tool and result. The talk, metaphors, diagrams are separate from the object, the ‘it’ of the end result being the different ‘readings’, just as the scaffold that supports a building as it is put up can be removed to leave the building ‘free-standing’.

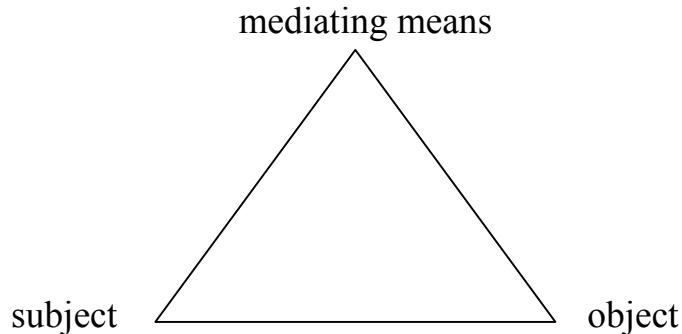


Figure 2

I want to question this separation of tool and result, by examining the nature of the outcome of this ‘lesson’ – what is the ‘basket’ that these teachers have ‘woven’? Where, exactly, do the ‘readings’ that emerge exist? The 2/5s, 2/3s and so forth, cannot be ‘there’ in the diagram itself (otherwise they would be immediately apparent). They cannot be solely ‘in’ the subjects’ heads waiting to be brought out. They cannot be ‘in’ the talk, metaphors and diagrams – these help to bring the readings into existence, but they are not the readings per se. So there is not an object – a ‘reading’ of the diagram – that exists ‘out-there’ waiting to be brought into ‘being’ through mediating means. The readings emerge and develop through the unity of the subject-mediating means-object. They are an example of tool-and-result.

Tool-and- result means that no part of the practice can be removed and looked at separately. Like the classic vase and faces optical illusion neither the faces nor the vase can be removed and leave the other. There are no ‘scaffolds’ or other mediating means in terms of metaphors or diagrams that can be ‘removed’ to ‘leave’ the ‘objects’ of instruction (fractional readings). The ‘practice of method’ is entire and ‘(t)he practice of method is, among other things, the radical acceptance of there being nothing social (-cultural-historical) independent of our creating it’ (Newman and Holtzman, 1997, p. 107).

The most important corollary of the ‘practice of method’ is, for Newman and Holtzman the priority of creating and performing over cognition:

We are convinced that it is the creating of unnatural objects–performances–which is required for ongoing human development (developing). (Newman and Holtzman, *ibid*, p. 109).

Taking as my starting point that school mathematics involves the ‘creating of unnatural objects’ – mathematical objects – I now present an example of what mathematics as performance might look like.

EXAMPLE OF A PERFORMATORY APPROACH TO MATHEMATICS.

This example comes from work in a school that had a history of ‘under-performance’ in National Test scores (The development through performance perspective raises questions about what it means to ‘under-perform’). When I started working in the school, the culture was such that the teachers spoke of the children not being able to ‘do’ mathematics (Again a ‘performance’ perspective raises questions. Clearly there was a lot that these children could do. Observing them in the playground they were as capable as any other children of being able to ‘do’ play and, for some, with both parents out at work, they were able to ‘play’ a variety of roles at home, including care-takers of siblings. What makes ‘doing’ mathematics any more difficult than being able to ‘do’ play in the school-yard or ‘do’ the preparation of a meal?)

In particular, the teachers spoke of the children not being able to talk about mathematics, and so the classroom environments were ones where the children were not encouraged to talk about mathematics.

The performatory approach adapted was one of creating environments where children were encouraged and expected to talk. Two factors were central to this: the use of engaging contexts that the children could mathematise (Freudenthal, 1973) and getting children to cooperate in pairs to develop (improvise) solutions and then pairs coming to the front of the class to ‘perform’ their solutions.

This example comes from working with a class of six- and seven-year-olds. It took place in February after we had been working with the class in this way since the previous September. The problem is adapted from the work of Fosnot and Dolk (2001). These situations were set up orally, not simply to reduce reading demands but to encourage children to ‘enter into’ the ‘world’ of the story. Most of the time the children willingly did this. Occasionally someone would ask ‘is this true’ (answered playfully with ‘what do you think?’) or a child would say, in a loud stage whisper ‘it’s not true you know, he’s making it up’, but even such ‘challenges’ to the veracity of the stories were offered and met with good humour and a clear willingness to continue to ‘suspend disbelief.’

The context set up was that I had gone to visit a cousin in the country, who ran a sweet shop. One of her popular lines was flavoured jellybeans. These were delivered in separate flavours and then mixed together for various orders. During my visit my cousin had some bags of six different flavours of jellybeans: did she have enough jellybeans to make up total order for 300 beans? Knowing that I was a teacher, she wondered whether the children I worked with would help her figure out if she had enough. Of course they would. I invited the children to offer flavours, hoping that they would come up with some Harry-Potteresque suggestions (ear-wax or frog?) for

flavours. They were conservative in their choices of fruit flavours, so I added in the last two. On the board was a list of flavours and the numbers of each: Strawberry – 72; Orange – 23; Cherry – 33; Apple – 16; Broccoli; 20; Fish – 72.

The class had been introduced to using the empty number line as a tool (prop in the theatrical sense) for supporting addition and subtraction, but they had not been presented with a string of five two-digit numbers before – this was well in advance of what children of this age are expected to be able to do independently. All the children were provided with

paper and pencil but we also had other props ready to hand in the form of base-ten blocks for anyone who wanted or needed to use these. Some did use the blocks but the majority of the class were content to work only with paper and pencil. Here are two improvised solutions from two pairs of children – a girl and boy in each case.

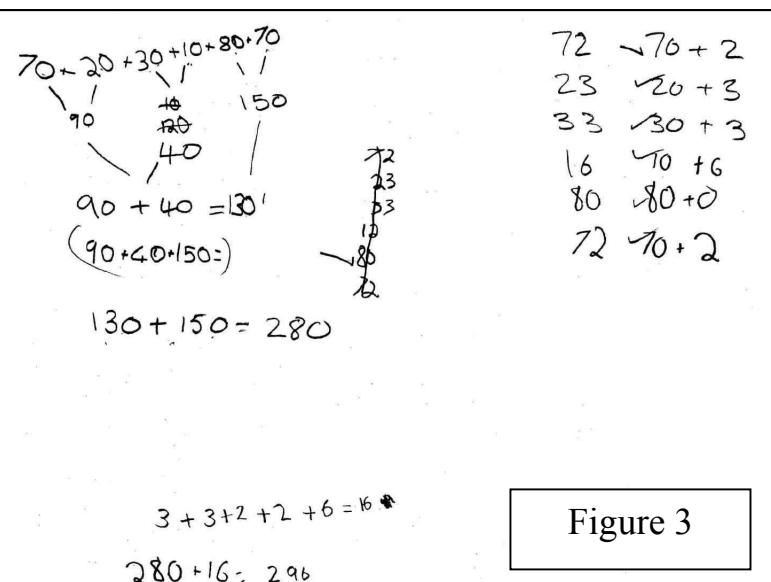


Figure 3

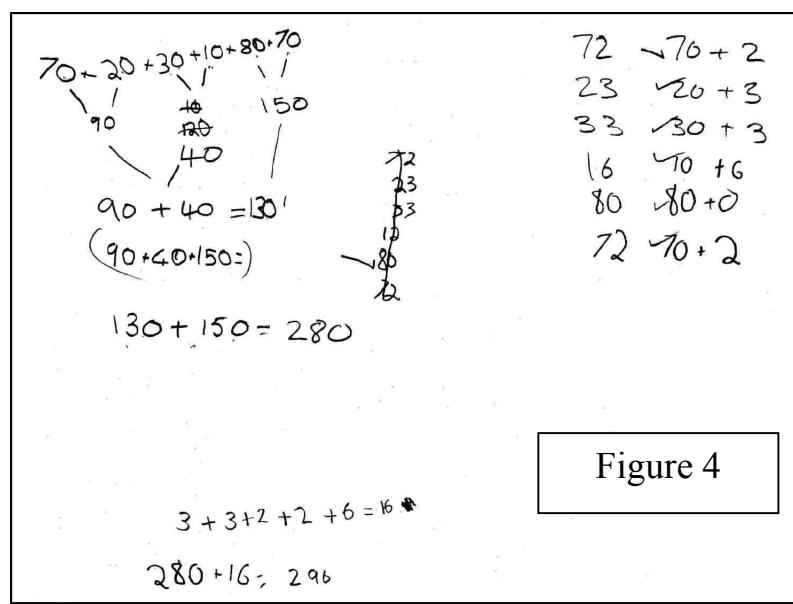


Figure 4

The children's whose work is shown in Figure 3 wrote down the six numbers in the order that they had been put up on the board, but added them in the order of largest number to smallest. They could figure out that $80 + 72$ was 152 without writing everything down, ticked off these two numbers and then used an empty number line to add on each of the remaining four numbers in descending order of magnitude.

The children's whose work is shown in Figure 4 adopted a different approach. They partitioned each number into its constituent tens and ones, added the tens, two at a time, until the total number of tens was reached. Then they added the ones, finishing off by adding the tens and ones together.

As the children were figuring out their solutions, the teacher and I were able to decide who would 'perform' their solutions to the rest of the class. These two pairs were

included in this selection, and, like the others chosen, were given due warning of this so that they had time to prepare what they were going to say

DISCUSSION

Did the children learn about addition through this lesson? I cannot say. What was of concern was that they learnt that mathematics is learnable and that they were capable of performing it. Developmental learning involves learning act as a mathematician and the realisation that the choice to continue to act as mathematicians is available. Developmental learning is thus generative rather than aquisitional. As Holtman (1997) problematises it:

Can we create ways for people to learn the kinds of things that are necessary for functional adaptation without stifling their capacity to continuously create for growth?

This is a key question for mathematics education. In England, and elsewhere, policy makers are specifying the content and expected learning outcomes of mathematics education in finer and finer detail. For example, the introduction of the National Numeracy Strategy in England brought with it a document setting out teaching and learning objectives – the ‘Framework for Teaching Mathematics from Reception to Year 6’ (DfEE, 1999) – a year-by-year breakdown of teaching objectives. The objectives within the framework are at a level of detail far exceeding that of the mandatory National Curriculum (NC).

The NC requirements for what 7- to 11-year-olds should know and understand in calculations is expressed in just over one page and, typically, include statements like:

work out what they need to add to any two-digit number to make 100, then add or subtract any pair of two-digit whole numbers, handle particular cases of three-digit and four-digit additions and subtractions by using compensation or other methods (for example, 3000 – 1997, 4560 + 998) (Department for Education and Employment (DfEE), 1999a, p.25).

In contrast, the Framework devotes over 50 pages to elaborating teaching objectives for calculation, at this the level of detail:

Find a small difference between a pair of numbers lying either side of a multiple of 1000

- For example, work out mentally that:

$$7003 - 6988 = 15$$

by counting up 2 from 6988 to 6990, then 10 to 7000, then 3 to 7003

- Work mentally to complete written questions like

$$6004 - 5985 = \quad 6004 - \quad = 19 \quad - 5985 = 19$$

(Department for Education and Employment (DfEE), 1999b, Y456 examples, p 46)

While teachers have welcomed this level of detail, there is a danger that ‘covering’ the curriculum (in the sense of addressing each objective) becomes the over-arching goal of teaching, that ‘acquisition’ of knowledge by learners becomes paramount and the curriculum content reified and fossilised. In particular the emphasis is on knowing rather than developing. Does ‘coverage’ of pages of learning outcomes help students view themselves as being able to act as mathematicians?

CONCLUSION

What might it mean to have a mathematics education that is not predicated on ‘knowing’ – after all, is not the prime goal that students should come to ‘know’ some mathematics. I think the issue here is not whether or not we should (or even could) erase ‘knowing’ from the mathematics curriculum, but that we need to examine carefully what it means to come to know, and in particular, the ‘myth’ that, as teachers, we can specify in advance exactly what it is that students will come to know. The scaffolding metaphor carries certain connotations – plans and blue-prints. In architecture, the final product, the building, can be clearly envisaged in advance of starting it. Thus ‘scaffolding’ appeals to our sense of being in control as teachers, it taps into a technical-rationalist view of teaching and learning. Get your plans carefully and clearly laid out (and checked by an authority, or failing that ‘download’ them from an authoritative source), put up the right scaffold in the lesson and all will be well (if only!).

Rather than learning in classrooms being built up in this pre-determined way, I want to suggest that maybe it is more like ants constructing ant-hills: Ants don’t (at least we assume) start out with a blue-print of the ant-hill that will be constructed. The ant-hill emerges through their joint activity. What emerges is recognizably an ant-hill (and not an eagle’s nest) although the precise structure is not determined until completion (if such a state ever exists). Ants are not ‘applying’ a method in order to construct ant-hills, they are simply practicing their method.

In the same way, children playing at being ‘mummies-and-daddies’ are not ‘applying’ a method of ‘play’ they are simply involved in the practice of play. Such play is performatory (they don’t sit around planning what to play, they simply get on and play) and improvisational (the ‘events’ and ‘shape’ of the play emerge through the practice of the play). The children’s play does not set out to be ‘about’ anything in particular, expect in the broadest terms of being about ‘mummies-and-daddies’ as opposed to, say, ‘princesses-and-princes’. It is this playful, performatory, improvisational practice of method, that Holtzman and Newman argue can help classrooms become developmental

In ‘playing’ the roles of helping the shopkeeper solve her problem, through cooperations the children were able to perform ‘beyond’ themselves, “performing a head taller than they are” (Vygotsky, 1978 p. 102) In such circumstances, the mathematics emerges in classrooms, but the precise nature of it cannot be determined in advance – I have to trust to the process, rather than try to control it. The solution methods to the jellybean problem could not have been closely pre-determined, but trust in the capability of the children to perform as mathematicians allowed rich solutions to emerge.

That is not to say that teaching does not rest on careful preparation – good improvisation does too – but that the unfolding, the emergence of a lesson cannot be

that tightly controlled (or if it is that the learning that emerges is limited and restricted to being trained rather than playing a part).

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HOW CAN WE ASSESS MATHEMATICAL UNDERSTANDING?

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In assessing students in mathematics, a problem we face is that we are all too often assessing only a limited part of their understanding. For example, when asking a student to carry out a multiplication calculation, are we really assessing their understanding of multiplication? To be clear about how we do this, we need to be clear about understanding itself. Therefore, this paper begins by providing an overview of what we mean by this concept of understanding. Having established a working definition, we examine a range of possible approaches that we can bring to assessing understanding in mathematics.. The contribution of this paper is to clarify this link between assessment and understanding, and explain why more novel methods of assessment should be used for this purpose,

The aim of this theoretical paper is to examine the different ways in which we can examine students' understanding in mathematics. In order to do so, we begin by defining exactly what we mean by understanding in this context, before moving on to examine what this means for the methods of assessment that we can employ. By being clear about what understanding is, we can show that more 'novel' approaches to assessment are needed if we are to try and access this. This will have implications for how we carry out research into students' understanding of mathematical concepts.

DEFINING 'UNDERSTANDING'

We begin by examining some definitions or explanations of 'understanding' in mathematics. Skemp (1976) identified two types of understanding; relational and instrumental. He described relational understanding as "knowing both what to do and why" (p. 2), and the process of learning relational mathematics as "building up a conceptual structure" (p. 14). Instrumental understanding, on the other hand, was simply described as "rules without reasons" (p. 2).

Nickerson (1985), in examining what understanding is, identified some 'results' of understanding: for example agreement with experts, being able to see deeper characteristics of a concept, look for specific information in a situation more quickly, being able to represent situations, and envisioning a situation using mental models. However, he also proposed that "understanding in everyday life is enhanced by the ability to build bridges between one conceptual domain and another" (p. 229). Like Skemp, Nickerson highlighted the importance of knowledge and of relating knowledge: "The more one knows about a subject, the better one understands it. The

richer the conceptual context in which one can embed a new fact, the more one can be said to understand the fact.” (p. 235-236) Hiebert and Carpenter (1992) specifically defined mathematical understanding as involving the building up of the conceptual ‘context’ or ‘structure’ mentioned above.

“The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of its connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections.” (p. 67)

Therefore, this idea of understanding being a structure or network of mathematical ideas or representations comes out clearly from the literature.

Another important issue that emerges from the above discussion is whether we are referring to understanding as an action or as a result of an action. Sierpinska (1994) clarified this by putting forward three different ways of looking at understanding. First of all, there is the ‘act of understanding’ which is the mental experience associated with linking what is to be understood with the ‘basis’ for that understanding. Examples of bases given by her were mental representations, mental models, and memories of past experiences. Secondly, there is ‘understanding’ which is acquired as a result of the acts of understanding. Thirdly, there are the ‘processes of understanding’ which involve links being made between acts of understanding through reasoning processes, including developing explanations, learning by example, linking to previous knowledge, linking to figures of speech and carrying out practical and intellectual activities. Sierpinska (1994) saw these processes of understanding as ‘cognitive activity that takes place over longer periods of time’ (p. 2). In making links between understandings of a mathematical concept through reasoning, for example showing why 12×9 gives the same answer as 9×12 , we further develop our understanding of the concept. Duffin and Simpson (2000) developed Sierpinska’s categories, referring to the three components as building, having and enacting understanding.

Drawing the various view points from past studies together therefore, the definitions of understanding that we use are the following:

- To understand mathematics is to make connections between mental representations of a mathematical concept.
- Understanding is the resulting network of representations associated with that mathematical concept.

These definitions draw together the idea of understanding being a network of internalised concepts with the clarification of understanding as an action and a result of an action. We have drawn on the definition of Hiebert and Carpenter (1992) and broadly termed what we link within this network (including the mental representation of what we are trying to understand, mental models and the memories of past experiences) as mental representations. We view the enacting of or the result of understanding as being distinct, which has implications later when we come to consider how we assess understanding in mathematics.

DEFINING ‘REPRESENTATIONS’

Having adopted a definition for understanding that involves ‘representations’, we should now define what we mean by this. First of all, we should clarify that we are referring to mental or internal representations, using the definition from Davis (1984):

“Any mathematical concept, or technique, or strategy – or anything else mathematical that involves either information or some means of processing information – if it is to be present in the mind at all, must be represented in some way.” (p.203)

By their very nature, we can only speculate on the possible forms of internal representations. In previously trying to obtain a definition of understanding, we grouped together the bases suggested by Sierpinska (1994) under the term ‘mental representations’. In addition though, Goldin (1998) put forward a variety of internal representations; verbal/syntactic, imagistic, symbolic, planning/monitoring/controlling and affective representation. An internal representation of a mathematical concept might therefore involve facts about that concept, pictures or procedures we might draw on in order to explore the concept, and how we have felt in the past working with that concept. In coming to understand more about that mathematical concept, we link together these separate representations to create a more complex understanding about that concept.

Although understanding in mathematics is based on internal representations, in instruction and assessment, what we actually use are ‘external’ representations of concepts. For example, external representations such as spoken language, written symbols, pictures and physical objects are used in order to communicate mathematics (Hiebert and Carpenter, 1992). Miura (2001) referred to internal and external representations as ‘cognitive’ and ‘instructional’ respectively. The processes of understanding put forward by Sierpinska (1994) can be mediated through external representations, for example the proof of why $\sqrt{2}$ is an irrational number can be achieved through the manipulation of symbols and logical statements. In assessment, we are usually asking students to communicate the result of their understanding, using external representations to do so. One might assume that there will be some connection between the external representations used in instruction or assessment, and the internal representations developed by students. However, we must recognise the difficulty that “it is more problematic to assume that the connections taught *explicitly* are internalised by the students.” (Hiebert and Carpenter, 1992, p. 86, our emphasis.) We can only try and access students’ understanding through external representations, in light of what we think is the structure of this understanding.

ASSESSING UNDERSTANDING

Having clarified what we mean by mathematical understanding, we can now finally turn to the main aim of the paper, that is to look at ways in which we can assess understanding in mathematics. We have already highlighted a drawback to any

potential method, using external representations of mathematical concepts to try and access connections made between internal representations. However, our clarification has resulted in two points which we can take forward:

- Understanding as connections made between mental representations.
- This understanding is distinct from the result (or enacting) of that understanding.

Hiebert and Carpenter (1992) stated that “understanding usually cannot be inferred from a single response on a single task; any individual task can be performed correctly without understanding. A variety of tasks, then, are needed to generate a profile of behavioural evidence.” (p. 89) In recognising understanding as a more complex network, if we are to assess this understanding then we need to try and access the different connections that a student has. All too often, when we assess students’ understanding in mathematics, we are gaining insight into only a small part of this network. Also, being able to carry out a mathematical task implies only that some understanding is there (this might only be links between the concept and a procedural model associated with that concept), not the extent of that understanding. An interesting development of this idea is the ‘non-binary nature’ of understanding (Nickerson, 1985). If a student has come across a concept in any way, then they will have some understanding of that concept, however limited or inappropriate the links within their understanding might be. Also, we can never have complete understanding; we can always develop understanding by developing more links, for example between apparently very different concepts that we have not associated together previously.

In light of these points, we can consider different ways in which we can assess this understanding. Possible methods suggested by Hiebert and Carpenter (1992) were to analyse:

- Students’ errors.
- Connections made between symbols and symbolic procedures and corresponding referents.
- Connections between symbolic procedures and informal problem solving situations.
- Connections made between different symbol systems.

We can use these suggestions as a starting point to examine the possibilities for assessing mathematical understanding.

Students’ errors

From our definition of mathematical understanding, we can see why simple calculations that we often use in the classroom are limited in their ability to assess understanding. For example, as highlighted by Skemp (1976), students can work instrumentally, only linking procedural representations with the concept and not other representations that might explain why the procedures are appropriate. Hiebert and

Carpenter (1992) state that “any individual task can be performed correctly without understanding.” (p. 89) We would modify this and say that an individual task can be performed with only limited understanding. Therefore, the fact that a student gets a calculation correct tells us little about the extent of their understanding. However, when a student makes a mistake in a calculation, then this might indicate the limitations of their understanding, even if that understanding is only instrumental.

Connections made between symbols and symbolic procedures and corresponding referents

Alternatively, if we ask students to explain what they are doing in mathematical tasks, then we can try and infer the links that they have made between different mental representations. Students might explain a calculation using external concrete objects, pictures or other symbolic procedures. Of course, we have the limitation that we are not directly accessing students’ internal representations. However, we might expect extensive links made between external representations to imply a more developed network of internal representations. As an example of this approach, Davis (1984) outlined the methodology for ‘task-based interviews’ used by him to examine students doing mathematical calculations. The students can be asked to ‘talk aloud’ whilst doing the calculation, with audio and video recording of the sessions, as well as the writing produced by the student and observation notes made by researchers. Providing students with opportunities to explain their reasoning in written tests can be used for this purpose as well. Watson *et al.* (2003) provided another interesting example where a questionnaire to examine students’ understanding of statistical variation was developed. Opportunities to ‘explain’ responses, as well as the straightforward stating of answers, were included in the questionnaire. All responses and explanations were coded according to the perceived level of understanding of the topic, and a Rasch analysis was carried out on the resulting data to obtain a hierarchy of understanding of different aspects of statistical variation. We can also use student errors for the purpose of eliciting these connections between representations. If we ask students to explain why an error has been made in a calculation, then what we are encouraging them to do is to show the links between the procedural representation and other representations for that concept. For example, Sowder and Wheeler (1989) based interviews around solutions to estimation problems from hypothetical students in order to elicit explanations.

Connections between symbolic procedures and informal problem solving situations

In analysing students’ approaches to solving specific tasks involving a particular mathematical topic, our examination of understanding may be constrained by the ‘closed’ nature of the task. If we provide more ‘open’ problem solving situations where what is required is not obvious, then we might access more of the understanding held

by a student. For example, Cifarelli (1998) used algebra word problems to access problem solving processes used by students (therefore looking at broader procedural representations for tackling mathematical problems) rather than a specific understanding of a mathematical topic. Once again, task-based interviews with audio and video recording were used for data collection. Another recent study employing this type of approach was carried out by Kannemeyer (2005). However, in analysing the resulting data, approaches to problem solving were categorised in terms of different categories of explanation provided by students. A ‘fuzzy’ mark (between 0 and 1) was awarded for each category in any given explanation, and the marks combined together using fuzzy logic to obtain an overall mark for understanding. This approach enabled quantitative measures of understanding to be obtained from more unstructured qualitative data sources.

Connections made between different symbol systems

The example that Hiebert and Carpenter (1992) provided for this form of assessing understanding was to examine extending the use of a particular symbolic representation (e.g. being able to extend decimal notation to include thousandths) or using different symbolic representations of the same mathematical idea (e.g. representing decimals as fractions). However, we can take a broader view of this category to include connections to common visual representations of mathematical ideas as well. For example, we can draw on some of our own work that we have recently been carrying out, looking at children’s understanding of multiplication. Harries and Barmby (2006) carried out research where children were asked to represent a multiplication sum using an array on a computer screen. Children worked in pairs with microphones connected to the computer, and a recording of their discussions and their work on the screen was made using Camtasia® software. This methodology provided us the opportunity of observing whether children could make the links between the symbolic and the particular visual representation in a more natural setting, without a researcher or video cameras possibly impacting on the situation. We plan to extend this research in the coming year, once again using computer programs to investigate children making links with a variety of visual representations and symbolic representation for multiplication. By looking at the variety of external representations that they can link together, we want to try and gain some insight into the internal understanding that they have for multiplication.

Other strategies for assessing understanding

As highlighted previously, in assessing the understanding of a mathematical concept, we want to find out the different internal representations that a student might hold about the concept, whether they are procedural, conceptual or even affective representations related to the concept. Also, we want to find out ways in which the representations are linked together. Two other strategies for trying to get an external manifestation for these internal links are to use *concept maps* and *mind maps*.

Brinkmann (2003) provided an overview for using these tools in the context of mathematics education, and Williams (1998) used concept mapping to assess students' understanding of the concept of function. McGowan and Tall (1999) also used concept maps produced at different times to show the development of students' understanding of function. Both concept maps and mind maps involve starting with a particular topic in a drawing, and placing ideas and concepts that one associates with the central concept in the drawing as well. Connections between these can be physically drawn as lines. Further ideas and concepts can then be added, emanating from the 'sub-concepts' that have been added around the central topic. Williams (1998) highlighted two possible advantages of using such methods: "The rationale for using concept maps in this study was to maximise participant involvement and to minimize the researcher's intrusive role ... Mathematical knowledge and structure do not lend themselves to simple categorizations, but they can be depicted well by concept maps." (p. 414) By using concept or mind maps, we can view the links that students can make externally (e.g. does a child have a visual representation for a multiplication calculation?) and therefore indirectly assess their understanding and also imply gaps that they might have in their understanding.

SUMMARY

In this paper, we have used the existing literature to obtain a working definition for mathematical understanding. Having established this definition, we have been able look at what this means if we want to assess students' understanding of mathematical concepts. We have outlined a range of possible assessment methods for mathematical understanding, most of which are quite different from the more 'traditional' methods of assessment that we use. The next step that we will take in our own research is to use the ideas and methods set out in this paper (in particular, we hope to take a detailed look at children's understanding of multiplication). The clarification of the concept of understanding, and the identification of methods that we might employ, is an important step in looking at students' understanding of mathematical concepts.

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THE DISCURSIVE CONSTRUCTION OF MATHEMATICAL THINKING: THE ROLE OF RESEARCHERS' DESCRIPTIONS

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A variety of perspectives on the nature and role of discourse in the teaching and learning of mathematics have been developed and applied in recent years. The conduct of research in mathematics education can also, however, be viewed from a discursive perspective. In this paper, I draw on discursive psychology, which has been described as an anti-cognitivist, anti-realist, anti-structuralist approach to discourse analysis and psychology. Based on this perspective, I examine discursive features of a research paper on mathematical thinking to argue that, within the mathematics education research community, researchers' descriptions of students' behaviour and interaction make possible subsequent accounts of mathematical thinking, rather than the other way around.

DISCURSIVE PERSPECTIVES ON THE TEACHING AND LEARNING OF MATHEMATICS

A variety of perspectives on the nature and role of discourse in social life have influenced research within mathematics education in recent years. Some research has drawn on sociological theories of interaction, including interactional sociolinguistics and symbolic interactionism to explore the social organisation of mathematics classroom discourse, highlighting, for example, the conventions and norms that arise (Yackel and Cobb, 1996). A related body of work has drawn on sociocultural theory to argue that, in mathematics, talk is 'almost tantamount to thinking' (Sfard, 2001, p. 13; Lerman, 2001). Such studies have, for example, attempted to trace the processes of socialisation through which students learn to use mathematical discourse and to do mathematics (e.g. Zack and Graves, 2001). Others have been more interested in the specific nature of interaction in mathematics classrooms. This work includes studies that draw on social-semiotic perspectives to explore, for example, the nature and role of mathematical texts and of intertextuality in mathematics education (e.g. Chapman, 2003). Similarly, others have emphasised the situatedness of mathematical meaning within classroom discourse (Moschkovich, 2003). Some researchers have turned to post-structuralism to examine the processes through which mathematics, teachers and students are positioned or constructed by mathematical discourses (e.g. Brown, 2001). More recently, discursive perspectives have led to new insights into the relationship between mathematics classroom interaction and wider political concerns, such as, for example, the role of different languages in multilingual settings (Setati, 2003).

In general, the various approaches summarised above have sought to understand different aspects of mathematical thinking (which, for this paper, I will use to also

encompass mathematical learning, meaning and understanding). These approaches generally highlight a central role for language, symbols and interaction in mathematical thinking. Conducting such research is also, however, a discursive process, involving, for example, the production and interpretation of various kinds of texts, such as tape-recordings of interviews, lesson transcripts or field notes. The discursive nature of this process has received less attention within the field of mathematics education. In this paper, I explore one particular aspect of the research process: the published research paper. To do so, I will draw on ideas developed in discursive psychology, which are summarised in the next section. These ideas highlight, amongst other things, the role of description in constructing both mind and reality. Taking one research paper (Sfard, 2001) as an example, I explore the role of description in the construction of mathematical thinking in published accounts of research. I conclude by discussing some possible implications for research in mathematics education.

DISCURSIVE PSYCHOLOGY

Discursive psychology (e.g. Edwards, 1997; Edwards and Potter, 1992) has been described as offering an anti-cognitivist, anti-realist, anti-structuralist account of the relationship between discourse and cognitive process, such as thinking, meaning or remembering [1]. In the context of research in mathematics education, these points have the following implications (for which I have drawn particularly on Edwards, 1997):

- *Anti-cognitivist*: entails a shift from a focus on ‘what happens in the mind’ (as an individual mental process) to how ‘what happens in the mind’ is done through discursive practice (as a socially organised process); thus, the nature of mathematical thinking or meaning, for example, are jointly produced through interaction.
- *Anti-realist*: reality is seen as being reflexively (and so relativistically) constituted through interaction. Thus, in any given situation, mathematics or mathematical cognitive processes are not pre-given, but are brought about through talk. Rather than mathematical meaning, for example, being pre-determined by words, symbols or diagrams, participants read such meanings into these things through their interaction.
- *Anti-structuralist*: following the preceding point, mathematical meaning and the organisation of mathematical interaction are situated, both in time and in place, emerging from preceding interaction, rather than in standard, predictable ways.

As Edwards (1997, p. 48) points out, this perspective is to some degree related to socio-cultural approaches to psychology. Both approaches recognise the central role played by social processes, culture and language in the development of the human mind. For much research influenced by sociocultural theory, however, the aim is to understand how the mind works, even if mind is constructed through participation in

society. Discursive psychology, by contrast, is more interested in how ideas like 'mind' are constructed in particular situations. The difference, Edwards argues, is broadly between ontological and epistemological concerns:

In discursive psychology, the major sense of 'social construction' is epistemic: it is about the constructive nature of descriptions, rather than of the entities that (according to descriptions) exist beyond them. (Edwards, 1997, pp. 47-48)

In this approach, therefore:

Mind and reality are treated analytically as discourse's topics and businesses, the stuff that talk is about, and the analytic task is to examine how participants descriptively construct them. (Edwards, 1997, p. 48)

In the case of research in mathematics education, concerns with mathematical thinking, for example, or the nature of mathematics, would be treated as discursive constructs. I have sought to use these ideas to analyse various examples of mathematics classroom interaction (e.g. Barwell, 2001). My aim was to understand how school students and teachers jointly constructed mathematical understanding, thinking and learning. In this paper, however, I am interested in how mathematical thinking is constructed through the research process itself, particularly in research publications. To facilitate this inquiry, in the next section I examine one published paper by Anna Sfard.

DESCRIPTION AND MATHEMATICAL THINKING: AN EXAMPLE

The paper I have selected (Sfard, 2001) concerns the relationship between discourse and mathematical thinking. The paper is interesting, in that it compares two ways of viewing mathematical thinking: the cognitivist, learning-as-acquisition approach and a communicative, learning-as-participation approach. Sfard sees these two approaches as complementary (p. 49). One strand within the paper involves the presentation and discussion of an exchange between a pre-service teacher and a 7-year old girl, reproduced below (see Sfard, 2001, p. 19). In what follows, I examine this strand of the paper from the perspective of discursive psychology.

Teacher:	What is the biggest number you can think of?
Noa:	Million.
Teacher:	What happens when we add one to million?
Noa:	Million and one.
Teacher:	Is it bigger than million?
Noa:	Yes.
Teacher:	So what is the biggest number?
Noa:	Two millions.
Teacher:	And if we add one to two millions?
Noa:	It's more than two millions.
Teacher:	So can one arrive at the biggest number?

Noa: Yes.

Teacher: Let's assume that *googol* is the biggest number. Can we add one to *googol*?

Noa: Yes. There are numbers bigger than *googol*.

Teacher: So what is the biggest number?

Noa: There is no such number!

Teacher: Why there is no biggest number?

Noa: Because there is always a number which is bigger than that?

In discussing this exchange, initially from a cognitivist perspective, Sfard writes:

Clearly, for Noa, this very brief conversation becomes an opportunity for learning. The girl begins the dialogue convinced that there is a number that can be called 'the biggest' and she ends by emphatically stating the opposite: 'There is no such number!'. The question is whether this learning may be regarded as learning-with-understanding, and whether it is therefore the desirable kind of learning. (Sfard, 2001, p. 19)

This paragraph is a description of what happened in the conversation. The description is plausible. Nevertheless, the description constructs various aspects of mathematical thinking on the part of Noa. In particular, Noa is constructed as being 'convinced' that there is a biggest number at the start of the conversation. Sfard describes Noa's penultimate contribution that there is 'no such number' as 'emphatic', also implying a degree of conviction. The use of 'emphatic' is linked to the use of an exclamation mark (!) in the transcript, adding to the reasonableness of the description. Noa is also described as having produced opposing statements. These statements are implicitly interpreted as a chronological shift, which is, in turn, called 'learning'. 'Being convinced' and 'learning' are aspects of mathematical thinking that are, however, *read into* the conversation through the description. By juxtaposing two of Noa's statements and describing them as opposites, the description makes possible, for example, the interpretation that learning has taken place.

Later in the same article, Sfard offers, by way of contrast, a more discursive perspective on the same extract:

...much of what is happening between Noa and Rada may be explained by the fact that unlike the teacher, the girl uses the number-related words in an unobjectified way. The term 'number' functions in Noa's discourse as an equivalent of the term 'number-word', and such words as hundred or million are things in themselves rather than mere pointers to some intangible entities. If so, Noa's initial claim that there is a biggest number is perfectly rational. Or, conversely, the claim that there is no biggest number is inconsistent with her unobjectified use of the word 'number': After all, there are only so many number-words, and one of them must therefore be the biggest, that is, must be the last one in the well ordered sequence of numbers...Moreover, since within this type of use the expression 'million and one' cannot count as a number (but rather as a concatenation of numbers), the possibility of adding one to any number does not necessitate the non-existence of the biggest number. (Sfard, 2001, p. 46)

Again, Sfard offers a plausible account, with the aim, in this case, of resolving the puzzle raised earlier, of how Noa comes to be ‘convinced’ of opposing ideas in the space of a short conversation. Again, however, Sfard’s description constructs various forms of mathematical thinking on the part of Noa. A key feature of the description is the idea that Noa is *interpreting* the word ‘number’ as ‘an equivalent of the term ‘number word’’. Based on this description, Sfard is able to provide a rational account of Noa’s utterances. Indeed, the later part of the above paragraph is devoted to setting out the linguistic and mathematical basis for that rationality, which amounts to a reading of Noa’s mathematical understanding in the earlier stages of the exchange. Thus, the nature of the description is intimately related with the argument that Sfard is pursuing. By setting out a particular version of what is happening in the conversation, Sfard makes available particular inferences about Noa’s (cognitive) interpretations, which in turn fit in with Sfard’s larger argument that the conversation represents an example of discursive conflict:

...both interlocutors seem interested in aligning their positions. The teacher keeps repeating her question about the existence of ‘the biggest number’, thus issuing meta-level cue signalling that the girl’s response failed to meet expectations. In order to go on, Noa tries to adjust her answers to these expectations, and she does it in spite of the fact that what she is supposed to say evidently does not fit with her use of the words *the biggest number*. (Sfard, 2001, p. 46)

The notion of discursive conflict stresses the clash of habitual uses of words, which is an inherently discursive phenomenon. In our present case [of Noa], we could observe a conflict between the two interlocutors’ discursive uses of the words ‘number’ and ‘bigger number’. While aware of the fact that the teacher was applying these terms in a way quite different from her own, Noa was ignorant of the reasons for this incompatibility. In this case, therefore, the girl had to presume the superiority of her teacher’s use in order to have any motivation at all to start thinking of rational justification for a change in her own discursive habits. (Sfard, 2001, p. 48)

The notion of discursive conflict originates, perhaps, in the first paragraph above, an account of the conversation in largely discursive terms; that is, in terms of cues, repeats and alignments. Even in this description, however, a degree of intention is read into Noa’s behaviour: she ‘*tries* to adjust her answers’, for example. This reading is then overlaid, however, with a more cognitively oriented account of ignorance, presumptions, motivations and thinking. Again, then, it is the nature of the description that makes possible the inferences about Noa’s mathematical thinking.

DISCUSSION

My purpose in examining Sfard’s paper in such detail is not to challenge her argument. Her exploration of the idea of cognitive conflict and her proposal of the alternative idea of discursive conflict are interesting developments and likely to be valuable for research and teaching. Rather, I am interested in how mathematical thinking is discursively constructed through the process of doing and communicating research in mathematics education.

My discussion of Sfard's paper particularly highlights the importance of *descriptions of mathematical behaviour* in constructing mathematical cognition. As Edwards (1997, pp. 37-43) has argued in the case of cognitive psychology, such descriptions play an important role: they make available particular interpretations of cognitive processes, whilst shutting out alternatives. Sfard's descriptions, for example, build in cognitive or discursive conflict, which can then be made explicit. More generally then, in written research reports, descriptions of mathematical behaviour are likely to be shaped to suit an author's wider argument concerning mathematical thinking and learning.

Of course many authors acknowledge their subjectivity, in the sense that they make it clear that their analyses are interpretations. Indeed Sfard, in her article, compares two such interpretations. Commonly, such acknowledgements are, however, based on the argument that, by using an explicit theoretical framework and giving sufficient detail about how the data were analysed, readers can make up their own minds about the trustworthiness of the analysis. This kind of argument is based on a desire to get at an objective cognitive process that can be interpreted, however indirectly and tentatively, through participants' behaviour. Students say things to their mathematics teachers and the task is to suggest what or how the student is thinking, learning or understanding. I am not making the reasonable point that language does not offer a 'window' into the mind. Research from a broadly socio-cultural perspective (including that of Sfard) recognises the complex role of language and culture in mathematical cognition and in the research process and generally accepts this point (see, for example, Lerman, 2001). Nevertheless, such research is attempting to say something about minds and their relationship with the world.

There are a couple relevant points, then, that are commonly accepted by qualitative researchers relevant to this issue. Firstly, it is accepted that no description can be perfectly accurate. Secondly, it is accepted that all descriptions include some degree of bias or subjectivity on the part of the author. Both these points, however, assume that the problem is, simplistically put, one of accuracy. My point is slightly different: it is that the descriptions themselves constitute mathematical cognition as it is theorised and conjectured about in publications about mathematics education. So Sfard's descriptions of Noa's conversation with the teacher make possible the ideas about cognitive and discursive conflict that Sfard wishes to discuss (as much as the other way around).

IMPLICATIONS AND QUESTIONS

Providing descriptions of the interaction or behaviour of people engaged in doing or learning mathematics is an integral part of much published research in the field. There are, however, particular ways of writing such descriptions that are specific to the kind of writing found in research papers, as opposed to, say, newspaper reports. Moreover, whilst I have highlighted the role of description in the construction of mathematical thinking in written research reports, other parts of the research process are likely to draw on other practices. Interviews, for example, can be seen to construct

mathematical thinking in particular ways. Similarly, discussions between researchers or informal conversations at PME meetings may also contribute in different ways to the construction of mathematical thinking. An interesting question then arises of how these different discursive practices work together to produce accepted accounts of mathematical thinking within a wider community. It is also interesting to consider how descriptions work in different ways at different points in the process. The transcript of Noa and her teacher, for example, can be seen as another form of description, perhaps produced prior to analysis and the writing of a research paper. The structuring of such a transcript is also implicated in the construction of mathematical thinking that arises, as with, for example, the role of the exclamation mark I pointed out earlier.

The practices I have highlighted are likely to be familiar to members of the PME community. PME members, however, are not the only people interested in mathematical thinking; it is also of interest to, for example, mathematics teachers, government advisors, students or textbook writers. The National Numeracy Strategy of England and Wales (DfEE, 1999), for example, includes descriptive lists of what 'pupils should be taught to know', thereby constructing mathematical thinking as something that can be listed and broken down into small components to which teachers can be held accountable. In this case, therefore, descriptions are used to construct mathematical thinking in rather different ways to those found in research papers. This observation leads to a more general question. If the discursive practices of mathematics education research are implicated in the construction of mathematical thinking within the community of researchers, and if different discursive practices are used to construct mathematical thinking in different ways in different communities (such as those of teachers of curriculum writers), what is the relationship between them?

Given the layers of reflexivity inherent in the argument I have made in this paper (including, for example, my use of descriptions of Sfard's work), it would not make sense to suggest, for example, that researchers should be more careful when producing their own descriptions. Every description in a paper concerned with mathematical thinking is implicated in its construction. Two courses of action are perhaps possible, each equally valuable. One is for researchers to be aware of the practices through which their accounts of mathematical thinking are produced. The other is for further research to uncover more of the practices of our community, so that that awareness can be enhanced.

NOTE

1. The three antis are derived from a response given by Margaret Wetherell as part of a UK Linguistic Ethnography Forum colloquium at the annual meeting of the British Association for Applied Linguistics, Bristol, 15-17 September 2005.

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AUTHORITY AND ESTEEM EFFECTS OF ENHANCING REMOTE INDIGENOUS TEACHER-ASSISTANTS' MATHEMATICS-EDUCATION KNOWLEDGE AND SKILLS¹

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The interaction between Australia's Eurocentric education and the complex culture of remote Indigenous communities often results in Indigenous disempowerment and educational underperformance. This paper reports on a mathematics-education research project in a remote community to support Indigenous teacher assistants (ITAs) in mathematics and mathematics tutoring in an attempt to reverse Indigenous mathematics underperformance. It discusses teachers' and ITAs' power and authority within school and community, describes the project's design, and summarises the project's results in terms of affects and knowledge. It draws implications on the relation between ITA professional development (PD), affect, esteem, knowledge, authority, teacher-ITA partnerships, and enhanced Indigenous mathematics outcomes.

For the last six years, we have been visiting remote communities to work with schools in an attempt to reverse the mathematics underperformance and low retention rate of Australian Indigenous students (MCEETYA, 2005; Queensland Studies Authority [QSA], 2006). We have found that a typical remote Australian Indigenous classroom has two staff members, namely, a young inexperienced non-Indigenous teacher and an older experienced ITA from the community (Cooper, Baturo & Warren, 2005; Warren, Cooper & Baturo, 2004). The teachers lack PD in Indigenous education and in working with another adult while the ITAs lack PD in how to assist the teacher educationally; the teachers usually leave the school after two years while the ITAs tend to remain. Thus, long-term projects with teachers in remote communities are problematic and led us to believe, like Clark (2000), that sustainable progress in remote school requires ITAs to be given a more central role in teaching.

ITAs find that Australia's highly Eurocentric education system (Rothbaum, Weisz, Pott, Miyake, & Morelli, 2000) lacks cultural understandings and clarity, leaving them with undefined roles and a sense of disempowerment even though they often hold positions of authority and esteem within their own communities (Matthews, Watego, Cooper & Baturo, 2005; Sarra, 2003). As a consequence, we find that most teacher-ITA interactions are impoverished and unjust (Sarra, 2003; Warren, Cooper & Baturo, in press) and fail to take account of the ITAs' strengths (Cooper, Baturo & Warren, 2005). For this reason, we have changed our research focus from teachers to ITAs. This paper analyses the first research project¹ we undertook with ITAs.

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POWER AND AUTHORITY IN SCHOOL AND COMMUNITY

Weber (Haralambos, Holborn & Heald, 2004) claims that *power* is realisation of will against the resistance of recipients, and *authority* is power legitimised by recipients.

He identifies three types of authority: (1) traditional authority which is the “taken for granted” or consensual authority given to a role (such as “teacher”); (2) charismatic authority which comes from the special personality qualities of authority figure; and (3) bureaucratic authority which is based on legal structures (such as schools) (Haralambos et al., 2004). Table 1 (extended from Warren, Baturo & Cooper, in press) summarises these types for non-Indigenous teachers (called Ts for the table) and ITAs within *school* and *community*. The classifications depend on non-Indigenous teachers’ and ITAs’ roles in these two very different social structures.

Table 1: *Authority types, Teachers and ITAs in school and community*

Authority type	School		Community	
	T	ITA	T	ITA
Traditional	✓	✗	✗ unless long-term or comm. role	? depends on family status
Charismatic	? depends on T	? depends on ITA	? depends on T	? depends on ITA
Bureaucratic	✓	✗ unless role part of school structure	✗ not present	? depends on comm. status

Commonly, the teacher’s authority comes from the school (traditional and bureaucratic authority), while the ITA’s authority comes from the community as most are respected community members or elders (predominantly traditional authority). This leaves many possibilities: (1) the most likely is for an ITA to have high traditional and, possibly, charismatic authority within the community, but no bureaucratic authority within the school; (2) an ITA with little community respect and poor charisma can have little authority in and out of a classroom; and (3) an ITA with strong community authority can have this authority transfer to the school.

We also take cognisance of Foucault’s (1991) notion that power is a relation in which *knowledge* has effect. This is supported by Warren, Baturo, & Cooper’s (in press) findings and Smith’s (2002) arguments claiming that education generally improves authority, particularly for race. Smith also argues that Black Americans have less authority because they have less training, accreditation and status attainment. This includes *job authority* (the authority most at risk in terms of race) which is especially psychologically rewarding because it brings status inside and outside the workplace and is related to job satisfaction, personal identity and self esteem (Ardler, 1993).

THE “TRAIN A MATHS TUTOR” RESEARCH PROJECT

The project’s aim was to develop ITAs as mathematic tutors for underachieving Indigenous students (Baturo & Cooper, 2006). It was a qualitative interpretive action-

research collaboration incorporating Smith's (1999) decolonising methodology exhorting "empowering outcomes with the secondary (Grades 8-12) and primary schools (Grades 1-7) and the local Council within a remote Aboriginal community in which we had worked for three years. Our hope was that the project would provide an educational (rather than the usual behaviour management role) for the ITAs and to reverse Indigenous students' mathematics underperformance.

Participants. Eleven ITAs volunteered for the project (7 primary, 4 secondary), representing almost all the ITAs from both schools, a huge personnel investment. They were "long-term, local residents, mostly women, who work part-time for modest wages ... often parents or grandparents of students" (Ashbaker & Morgan, 2001. p. 2). Most feared mathematics, had received little PD in its teaching, and lacked understanding (only one could fully understand 3-digit numbers at the beginning of PD); their role in classrooms had been behaviour control. The PD took place in community buildings – PD sessions in a brand new council training building; lunch at the community centre. The tutoring trials took place at the secondary school with Years 8-10 students who lacked understanding of 3-digit numbers.

The Aboriginal community in which the research was sited was established in the early 20th century and set up by forcible removal of Aboriginal people from their traditional lands, many of which were more than 1000 km away. Until the late 1970s, the community was owned by the government and run by white staff. At the end of the 1970s, without training or preparation, the community was given to the Aborigines to run through a council. The community is made up of more than 10 different cultures making it difficult to get consensus on many issues. It shares the common problems of remote Indigenous communities: poverty, substance abuse, violence, poor health, low life expectancy and incarceration (Fitzgerald, 2001).

In the year before the project, the students' mathematics performance was below that of other similar Indigenous communities. School attendance was < 30%, behaviour was out of control in most classrooms, all Grade 2 students failed to meet State minimal standards, and many secondary students could not meet Grade 3 standards. The young inexperienced non-Indigenous teachers taught white urban mathematics with little or no Indigenous contextualisation (Matthews et al., 2005) to give relevance and build pride. However, the new primary-school principal was just about to set up a school renewal program based on the successful methods of Sarra (2003).

The PD program. The program was based on two main assumptions: (1) ITAs, being long-term community members, would be familiar with community mores and language, able to meet the students' cultural needs, and the schools' key to stability (Bature, Cooper & Warren, 2004; Clark, 2000), and (2) the PD should be "2-way strong" (we hoped to learn, from the ITAs, about Indigenous contextualisation and language in which to embed teaching). It was developed from three clusters of principles (Bature & Cooper, 2006): (1) *mathematics/pedagogy* – teaching for structural understanding (Sfard, 1991) using kinaesthetic learning with materials and developing informal and formal language (Bature, 2003, 2004); (2) *PD* - using train-

trial cycles (where ITAs trial their ideas with students) with just-in-time support and reflection (Baturo, Warren & Cooper, 2004) and enough time set aside to do all at the detail required; and (3) *social* – experiencing success (Clarke & Hollingsworth, 2002), building group cohesion and ITAs' identities (Sfard & Prusack, 2005) as tutors; and working in a positive learning space (Skill & Young, 2002).

Procedure. A mathematics-education manual was produced for the ITAs (see online report – Baturo & Cooper, 2006). The PD program ran four hours a day Monday to Thursday for 4 weeks. Week 1 provided the mathematics and pedagogic background for the tutors while Weeks 2 to 4 focused on providing the tutors with particular mathematics and pedagogic skills for crucial transition points in teaching whole-number numeration. Weeks 2 to 4 were organised so that, every second day, the aides could trial, with actual students at risk with respect to mathematics, the numeration activities they had learnt the previous day. Even though the ITAs knew us very well, they were reluctant to attend as (they later told us) they were nervous of "doing the big maths". However, once they had experienced a session, they attended regularly. Lunch at the community centre and reflection sessions were made social occasions where ITAs and researchers could build personal relationships and group cohesion. The material to be trialled with students was designed to motivate and to ensure successful learning.

The PD sessions, tutoring trials and reflection sessions were videotaped and field notes were written. At least 4 researchers were present for each PD and trialling session –one researcher taught while the others observed (PD) but all were available for intervention, if required, during the trials. Although some informal demographic data were gathered prior to the PD, we felt that we could not risk full pre-interviews with the ITAs because of the fragility of early attendance. We were also unable to undertake individual post-interviews with ITAs due to the remoteness of the site and the difficulty in organising times. However, we were able to undertake a collective post-interview.

Analysis. We evaluated the project from observations of, and informal discussions, with the ITAs during PD, trials and reflections, and audiotaped follow-up interviews with teachers and principals. In particular, we used a 5-point rating scale (1 low and 5 high) to jointly assess (all ratings were negotiated between 2 researchers) the ITAs at the beginning and end of the program in terms of: (1) mathematics and tutoring affects and beliefs, and (2) mathematics and pedagogy knowledge, and tutoring skill

RESULTS

Attendance and empowerment. The project was successful in terms of these ratings (Baturo & Cooper, 2005). Attendance was 90% for the PD when special personal and contextual circumstances were taken into account, a rate which experienced members of the community said was very high. For us, it was gratifying considering that attendance was not compulsory and PD took most of the school day (and some ITAs were under pressure from teachers to spend time in classrooms). One implication of such high attendance was that the PD "hit the spot" for ITAs in its focus (early mathematics), its pace and its stance that we were equal collaborators (2-way strong).

As their confidence increased across the four weeks, so too did their sense of job authority (as evidenced in their interactions with the students).

Knowledge, affects and empowerment. From researcher, teacher and student observations, the project was highly successful (Baturo & Cooper, 2006). All researchers and observers commented on how dramatically the ITAs had changed across the four weeks of the course as initial shyness had been replaced by, as one researcher called it, *a thirst for knowledge*. The ITAs wanted all that we could give them on the structure of the number system and techniques for teaching it. The feedback from the ITAs to the PD and tutoring was infectious; again all researchers and observers felt it.

The ITAs were excited about what they were learning and very proud of how they tutored the students; we were so nervous at the tutoring trials and the ITAs were so calm; the teachers and students were delighted by the tasks and their "new" tutors. We were also impressed with the ITAs' confidence, their use of materials and their questioning. One observer remarked that it was *like a dam had burst and there was a pouring out of interest*; another said it was *like rain falling on land after a drought*. One principal stated, *the ITAs have been more successful in their work with students and far more confident in the way they deal with students generally in the classroom, the kids have also responded positively*.

The teachers were amazed at what the ITAs could do in the trials and wanted them to return immediately to the classroom and repeat it with their students. One teacher commented, *kids were well behaved because they knew and liked what they were doing ... for kids who generally didn't succeed at school work, they liked the fact that they were able to succeed*, and another supported, *kids also seemed to enjoy every part of it particularly the one-on-one teaching they received*. The teachers also commented on how the ITAs had changed across the four weeks, saying that now they moved around the classroom helping students in all subjects, not sitting back and watching as they had done before. The most powerful evidence was from the primary school principal who later gave credit to her ITAs' tutoring skills for six students meeting the Year 2 minimum standards in mathematics, something which no student had achieved previously in the school.

However, it was the community's response to the PD that was most unexpected. The graduation ceremony we gave the ITAs became a major event in the town; many elders and community members attended as did external dignitaries in Indigenous Affairs. All applauded the success of the ITAs. One Elder commented that her generation was not educated past Year 4 and that she *never thought she'd see the day when people of her community would have the opportunity to undertake a university program*. Another commented that there was *no shame*. These and others indicated their pride in the ITAs' achievement.

Improvement in performance ratings was also significant (see Table 2), but this was almost a self- fulfilling prophecy considering the ITA's lack of prior PD and limited previous education, and reactions to this PD opportunity. The initial ratings indicate

the novelty of the PD (and nervousness of the ITAs) whilst the final ratings indicated that the ITAs had engaged, learnt and grown confident.

Table 2: IEWs' pre-post ratings for mathematics and mathematics tutoring affects, knowledge and tutoring skill

Affect, knowledge and tutoring-skill characteristics	Mathematics		Mathematics Tutoring	
	Pre Mean	Post Mean	Pre Mean	Post Mean
<i>Affects</i>	Motivation	1.6	4.5	1.7
	Confidence	1.6	4.4	1.5
<i>Knowledge</i>	2-digit numbers	2.5	4.5	1.1
	3-digit numbers	2.0	4.1	1.0
<i>Tutoring</i>	2-digit numbers	N/A	N/A	1.9
	3-digit numbers	N/A	N/A	1.7

Note. N/A means "Not Applicable".

DISCUSSION AND IMPLICATIONS

From the point of view of everyone involved, the PD program was successful – it improved ITAs' mathematics knowledge and pedagogy and tutoring skills and built their confidence and gave them esteem within the community. It appeared to affect positively all who came in contact with it; for example, the government representative at graduation supported further grant applications and the primary principal's open support in a meeting of State principals secured us the school support we needed for follow up research. It was evident that the ITAs were amenable to, needed, and benefited from the PD and that the project's PD program was effective and efficient (Bature & Cooper, 2006). However, the interest in this paper is to understand how and why things appeared to work so well. For this, we will initially look at the effect of some of the program's underlying principles, then at authority, and finally at a serendipitous confluence of interventions. Finally we will draw implications for future PD and research.

Principles. The basis of the project's success appeared to be how the principles interacted. First, our choice of using council buildings rather than the school as the learning space appeared to be influential. It made the program overt, public and visible to the community who gathered each day at the shops and council offices. It also gave the ITAs and the project the appearance of council approval and boosted the ITA's esteem in the community. Second, our decision to teach mathematics for structural learning was also influential. The ITAs recognised that they were successfully learning and tutoring important basic concepts and processes (what they called, *the big maths*) not just simple facts and skills, and that the researchers had high expectations of them (an important component of improving performance and building pride according to Sarra, 2003). Third, our decision to have student trials and to design instruction to maximise ITA tutoring success worked, and boosted the ITAs' confidence, motivation, pride and commitment and made them willing to try other new things (a PD cycle described in Clarke & Hollingsworth, 2002).

Authority. The project's success can also be seen in terms of power and authority (see Table 3). The project provided the ITAs with knowledge and skills to operate successfully in classrooms in ways they had not known before, thus boosting ITA power and authority within the schools (Foucault, 1991; Smith, 2002). Similarly, in Weber's terms (Haralambos et al., 2004), the public nature of the project and the visible support of the council boosted the ITAs' *charismatic* and *bureaucratic* authority in the community, particularly as regards job authority (Ardler, 1993). Consequently, the balance of teacher-ITA authority changed positively for the ITAs as per Table 3.

Serendipity. At the same time as our project was running, the primary-school principal was putting in place a school renewal based on Sarra (2003), one of whose tenets is to increase Indigenous leadership in schools. As a consequence, the ITAs role in the primary school was given *bureaucratic authority*, thus further changing authority relationships (see Table 3). Thus, teacher-ITA partnerships were more equitable in primary than secondary school (which explains the greater success of the primary ITAs).

Table 3: *Authority types, Teachers and ITAs in school and community after PD*

Authority type	School		Community	
	T	ITA	T	ITA
Traditional	✓ present	✗ not present	✗ not present	? depends
Charismatic	? depends	✓ present	? depends	✓ present
Bureaucratic	✓ present	✗ not present ✓ present *	✗ not present	✓ present

Note. * represents the change in the primary school after the principal's school renewal.

Implications. Two main implications emerged from the projects' results. First, successful ITA PD requires structural mathematics, train-trial cycles, careful selection of learning space and a focus on success. These principles interact to build affect and esteem, and the ITAs become effective tutors and improve students' mathematics outcomes (Baturo & Cooper, 2006). Second, any PD program to educate ITAs will have effects on authority within school and community which in this project led to more balanced power relationships, better teacher-ITA partnerships, greater school Indigenous influence and leadership, more contextualisation, pride and belief in ability, and improved mathematics outcomes (Foucault, 1991; Matthews et al., 2005; Sarra, 2003; Smith, 2002; Warren Baturo, et al., in press). These authority changes must be catered for or they may have unwitting negative effects as well as serendipitous positive effects.

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RAISING STUDENTS' UNDERSTANDING: LINEAR ALGEBRA

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This study is a contribution to the ongoing research in undergraduate mathematics education, focusing on linear algebra. It is guided by the belief that better understanding of students' difficulties leads to improved instructional methods. The questions posed in this study are: What is students' understanding of linear (in)dependence? What can example-generation tasks reveal about students' understanding of linear algebra? This study identifies some of the difficulties experienced by students with learning the concepts of linear dependence and independence, and also isolates some possible obstacles to such learning. In addition, this study introduces learner-generated examples as a pedagogical tool that helps learners partly overcome these obstacles.

BACKGROUND AND THEORETICAL PERSPECTIVE

There is a common concern expressed in the literature that students leaving a linear algebra course have very little understanding of the basic concepts, mostly knowing how to manipulate different algorithms. Carlson (1993) stated that solving systems of linear equations and calculating products of matrices is easy for the students. However, when they get to subspaces, spanning, and linear independence students become confused and disoriented: "it is as if the heavy fog has rolled in over them." Carlson (1993) further identified the reasons why certain topics in linear algebra are so difficult for students. Presently linear algebra is taught far earlier and to less sophisticated students than before. The topics that create difficulties for students are concepts, not computational algorithms. Also, different algorithms are required to work with these ideas in different settings.

Dubinsky (1997) pointed out slightly different sources of students' difficulties in learning linear algebra. First, the overall pedagogical approach in linear algebra is that of telling students about mathematics and showing how it works. The strength, and at the same time the pedagogical weakness, of linear situations is that the algorithms and procedures work even if their meaning is not understood. Thus, students just learn to apply certain well-used algorithms on a large number of exercises, for example, computing echelon forms of matrices using the Gaussian row elimination method. Secondly, students lack the understanding of background concepts that are not part of linear algebra but important to learning it. Dorier, Robert, Robinet, and Rogalski (2000) identified students' lack of knowledge of set theory, logic needed for proofs, and interpretation of formal mathematical language as being obstacles to their learning of linear algebra. Thirdly, there is a lack of pedagogical strategies that give students a chance to construct their own ideas about concepts in linear algebra.

Examples play an important role in mathematics education. Students are usually provided with examples by teachers, but are very rarely faced with example-generation

tasks, especially as undergraduates. As research shows (Hazzan & Zazkis, 1999; Watson & Mason, 2004), the construction of examples by students contributes to the development of understanding of the mathematical concepts. Simultaneously, learner-generated examples may highlight difficulties that students experience. This study examines how and in what way example-generation tasks can inform about and influence students' understanding of linear algebra. The APOS theoretical framework was adopted in this study to interpret and analyse students' responses (Asiala et al, 1996).

METHOD

The participants of the study were students enrolled in Elementary Linear Algebra course at a Canadian University. The course is a standard one-semester introductory linear algebra course. It is a required course not only for mathematics majors but also for students majoring in computing science, physics, statistics, etc. 113 students participated in the study. Later in the course the students were asked to participate in individual, clinical interviews. A total of six students volunteered to participate in the interviews. These students represented different levels of achievement and sophistication.

The data for this study comes from the following sources: students' written responses to the questions designed for this study, and clinical interviews. To follow the example-generation process, Task: Linear (in)dependence was included in the interview questions as well. Having students generate examples and justify their choices through written responses and in an interview setting provided an opportunity not only to observe the final product of a student's thinking process but also to follow it through interaction with a student during his/her example-generation.

Task: Linear (in)dependence

- a). (1). *Give an example of a 3x3 matrix A with real nonzero entries whose columns a_1, a_2, a_3 are linearly dependent.*
- (2). *Now change as few entries of A as possible to produce a matrix B whose columns b_1, b_2, b_3 are linearly independent, explaining your reasoning.*
- (3). *Interpret the span of the columns of A geometrically*
- b). *Repeat part a (involving A and B), but this time choose your example so that the number of changed entries in going from A to B takes a different value from before.*

The prerequisite knowledge for many concepts in linear algebra is the linear dependence relation between vectors. The purpose of the task was to investigate students' understanding of the concept of linear dependence and linear independence of vectors, in particular, in \mathbf{R}^3 . Many concepts of linear algebra are connected, and students should be able to use all these terms freely and with understanding. On one hand, this task connects the number of linearly independent columns in a matrix A , the number of pivots in an echelon form of A , and the dimension of the vector space

spanned by the column vectors of A . On the other hand, it connects the minimum number of entries required to be changed in A to make its columns linearly independent, and the number of free variables in the matrix equation $A\mathbf{x} = \mathbf{0}$. This task also explores the possible proper subspaces of a vector space \mathbf{R}^3 (excluding the subspace spanned by the zero vector, $\text{Span}\{\mathbf{0}\}$). It can be further extended to a 4×4 case, and then to the general case of $n \times n$ matrices.

This is an open-ended task with no learnt procedures to accomplish it. The routine tasks ask students to determine if a set of vectors is linearly dependent or independent by applying the definition or theorems presented in the course. In part (a1) of the task, the given and the question are reversed. Zazkis and Hazzan claim that ‘such “inversion” usually presents a greater challenge for students than a standard situation’ (1999, p.433). To complete Task (a1) students have to adjust their prior experiences in order to construct a set of three linearly dependent vectors in \mathbf{R}^3 , viewed as columns of a 3×3 matrix A .

RESULTS: LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

The two concepts of linear dependence and independence are closely connected. To have a solid understanding of one of them involves having understanding of the other. I first present the summary of students’ responses for constructing matrix A with linearly dependent columns, and then use APOS theoretical framework to analyze students’ understanding of linear (in)dependence.

Constructing matrix A with linearly dependent columns

In Task (a1) the students were required to give an example of three linearly dependent vectors represented as a 3×3 matrix with nonzero real entries. Table 1 presents the summary of different approaches used to complete this part of the task. The total frequencies exceed the number of participants as some students provided two examples for the task. Although all but 6% of the students constructed correct examples, their methods indicate different levels of understanding.

The responses to the remaining parts of the task depended on the construction of a matrix A . The results and analysis of these remaining parts are presented with examples of students’ work below.

Linear dependence as action

For students using a guess-and-check strategy the linear dependence was concluded as an outcome of an action performed on a chosen 3×3 matrix A . To complete Task (a1), these students had to pick 9 numbers to perform a set of operations on these numbers getting a certain result, in this case, at least one zero row in the modified form of A . This is a consequence of the condition for the columns of a matrix to be linearly dependent. These students had to go through calculations explicitly to verify that their example satisfied the requirement of the task.

Table 1: Constructing 3x3 matrix A with linearly dependent columns

Method	Examples	Frequency of occurrence
Guess-and-Check method	$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}$	17%
Rows method: same rows, one row multiple of another row	$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 2 & 5 & 1 \end{bmatrix}$; or $A = \begin{bmatrix} 4 & 6 & 9 \\ 8 & -7 & -5 \\ 2 & 3 & 9/2 \end{bmatrix}$	23%
Echelon method: start with echelon form U of A	$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \rightarrow R3+R2+R1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\xrightarrow{R1 \rightarrow R1+R3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = A$	5%
Identical columns method: [$\mathbf{a}_1 \mathbf{a}_1 \mathbf{a}_1$] where \mathbf{a}_1 has nonzero real entries	$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$; or $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	8%
Multiple columns method: two columns are multiples of the first one - [$\mathbf{a}_1 \mathbf{c}\mathbf{a}_1 \mathbf{d}\mathbf{a}_1$] where \mathbf{a}_1 has nonzero real entries and c and d are both nonzero real numbers	$A = [\mathbf{a}_1 \ 3\mathbf{a}_1 \ 9\mathbf{a}_1] = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 9 \\ 1 & 3 & 9 \end{bmatrix}$; $A = [\mathbf{a}_1 \ 2\mathbf{a}_1 \ 3\mathbf{a}_1] = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$	19%

<p>Two multiple columns method: two identical columns or two columns multiples of each other and the third column having any nonzero real entries: $[\mathbf{a}_1 \mathbf{c}\mathbf{a}_1 \mathbf{a}_3]$</p>	$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}; \text{ or } A = [\mathbf{a}_1 \ 2\mathbf{a}_1 \ \mathbf{a}_3] = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 6 \\ 4 & 8 & 1 \end{bmatrix}$	21%
<p>Linear combination method: any two columns, \mathbf{a}_1 and \mathbf{a}_2, having nonzero real entries and $\mathbf{a}_3 = c\mathbf{a}_1 + d\mathbf{a}_2$</p>	$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_1 + \mathbf{a}_2] = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{bmatrix};$ $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ 2\mathbf{a}_1 + (-1/2)\mathbf{a}_2] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 1/2 & -1 & 3/2 \end{bmatrix}$	18%

Linear dependence as process

Applying the APOS theoretical framework, students are operating with the process conception of linear dependence when they construct a matrix A emphasizing the relations between the rows. They may know that in order for the columns of A to be linearly dependent an echelon form of a matrix has to have a zero row. The row reduction process is an intended action in this case. It is performed mentally, and then reversed to generate a required matrix.

There is an intermediate step that links the linear dependence of columns of a matrix and its echelon form having a zero row. The definition of linear dependence of a set of vectors is given in terms of a solution to a vector equation. That is, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if the vector equation, $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = \mathbf{0}$ has a nontrivial solution. The solution set of this vector equation corresponds to the solution set of a matrix equation $A\mathbf{x} = \mathbf{0}$ having the \mathbf{v}_i 's as columns which in turn corresponds to the solution set of the system of linear equations whose augmented matrix is $[A \ \mathbf{0}]$. In the prior instruction it was shown that the linear system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if it has free variables, and this can be inferred from an echelon form of A . Thus, some students formed the connection: linear dependence \leftrightarrow free variables \leftrightarrow zero row in echelon form. As a result some examples were justified with the following statements: 'a linearly dependent matrix is a matrix with free variables', 'columns of A are linearly dependent since x_3 is free variable which implies $A\mathbf{x} = \mathbf{0}$ has not only trivial solution', or 'when the forms are reduced into reduced echelon form, the linearly dependent matrix has a free variable x_3 ; however, the linearly independent doesn't – it has a unique solution'.

Only 50% of the students completed both parts of the task, with 63% of incorrect responses to Task (b). In the majority of incorrect responses students ignored the different structures of linear dependence relations between vectors. They either used the same matrix A in both parts of the task or a matrix A having the same linear dependence relations between columns. Then if the students changed the correct number of entries in Task (a), their response to Task (b) was incorrect. For instance, one student constructed matrix A for both parts with the same dependence relation between columns, \mathbf{a}_3 in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$, and changed 1 entry in the first part but 3 entries in the second part.

Linear dependence as object

The row reduction process is central to linear algebra. It is an essential tool, an algorithm that allows students to compute concrete solutions to elementary linear algebra problems. However, encapsulation of linear dependence as an object requires a movement beyond the outcome of actual or intended procedures of row reduction toward a conceptual understanding of the structure of linear dependence relations in a set of vectors.

An indication of the construction of linear dependence as an object is demonstrated when students emphasize the relation between vectors, when they use the linear combination method to construct their examples of three linearly dependent vectors. In the linear combination method, there could be recognized different levels of generality for constructing an example. Either students gave a specific example of a matrix with a linear dependence relations between columns easily identified, as can be seen in Table 1, or they identified a general strategy for constructing a class of 3x3 matrices with linearly dependent columns. For example, Amy wrote: 'to be linearly dependent, at least one of the columns of a matrix A has to be a linear combination of the others ... $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$ with weights not all zero. Pick \mathbf{a}_1 and \mathbf{a}_2 . Then for $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ to be linearly independent, \mathbf{a}_3 has to be a linear combination of \mathbf{a}_1 and \mathbf{a}_2 . So, let $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ '. In the latter case, students applied the property that if \mathbf{u} and \mathbf{v} are linearly independent vectors in \mathbf{R}^n , then the set of three vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent if and only if \mathbf{w} is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ (i.e. \mathbf{w} is a linear combination of \mathbf{u} and \mathbf{v}).

During clinical interviews, students were able to move from the process understanding of linear dependence to the object level. Initially, both Anna and Leon used matrix A with the same linear dependence relation between columns to complete Task (b). They were changing different number of entries but knew that some changes were unnecessary.

Working through this task helped students understand the connection between the linear dependence relations, the geometric interpretation, and the minimum number of entries needed to change:

Leon: ...actually, I think to make two changes is minimum, because all three vectors are linearly dependent to one another. Changing one will not change the relationship overall. You will still have at least two linearly dependent vectors. So can I draw

from that with three linearly dependent vectors you need two changes and with only two linearly dependent vectors you only need one change.

Anna even attempted to generalize her strategy for an nxn case:

Anna: First, if my vectors are the same it's going to take more than one step to make them linearly independent. But if two of the vectors are different and the last one is the same as one of the other ones, I just need to change the leading entry number in that matrix; so when I row reduce it, I have an identity matrix...If I have an n by n matrix, and I have $\{v_1, \dots, v_{n-1}\}$ and then I have $2v_1$. This one vector is twice v_1 , or three times v_1 , just to keep it general, as my v_n . So if I make $\{v_1, \dots, v_{n-1}\}$ linearly independent, and the very last one is cv_1 then I need to change only one entry.

Object conception of linear dependence relation includes mastery of all possible characterizations of a linearly dependent set of vectors, in particular, the ability to recognize the possible ways to alter a set in order to obtain a linearly independent set. In Task: Linear (in)dependence, encapsulation of linear dependence as an object includes viewing a matrix as a set of column vectors, not as discrete entries that have certain values after performing algebraic manipulations. The latter perspective inhibits students' geometric interpretation of the span of columns, because the structure of linear dependence relations is not visible. Thus, the students that correctly completed both parts of the task might be operating with the object conception of linear dependence.

CONCLUSION

In general, example-generation tasks provide a view of an individual's schema of basic linear algebra concepts. Through the construction process and students' examples we see the relationships between the different concepts. Task: Linear (in)dependence revealed that the connections linear dependence \leftrightarrow free variables / pivot positions / zero row in echelon form, and linear independence \leftrightarrow no free variables / vectors not multiples of each other are strong in students' schema.

Learners' responses to Task: Linear (in)dependence showed that many students treat linear dependence as a process. They think of linear dependence in reference to the row reduction procedure. Some students connected linear dependence to the homogeneous linear system $Ax = \mathbf{0}$ having free variables that in turn corresponds to the nxn matrix A having a zero row in an echelon form. Other students linked the linear independence of vectors to a homogeneous linear system having only basic variables and therefore n pivot positions. However, few students considered the different structures of the linear dependence relations.

Even though geometric representation helps in visualizing the concepts, for some students geometric and algebraic representations seem completely detached. This can be seen in students' attempts to provide a geometric interpretation of the span of the columns of a matrix. There was a common confusion of the span of the columns of A with the solution set of $Ax = \mathbf{0}$. Instead of providing a geometric interpretation of the span of the columns of A , some students gave a geometric interpretation of the solution set of the homogeneous system $Ax = \mathbf{0}$.

The tasks soliciting learner-generated examples were developed in this research for the purpose of data collection. However, these tasks are also effective pedagogical tools for assessment and construction of mathematical knowledge, and can contribute to the learning process. Part of the power of Task: Linear (in)dependence is that it anticipates the concept of rank, long before students are exposed to it. In playing with the examples (assigned after only two weeks of classes), students develop their intuition about what linear (in)dependence “really means”. The students may not be able to articulate why the second example works differently from the first, but they are starting to develop a “feel” for the difference. This task can be further extended to higher dimensional vector spaces.

It is hoped that by examining students’ learning, the data collected can lead to teaching strategies, which will help students expand their example spaces of mathematical concepts and broaden their concept images/schemas. It is proposed that further studies could discuss the design and implementation of example-generation tasks intended specifically as instructional strategies, and evaluate their effectiveness.

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INFORMAL CONCEPTIONS OF DISTRIBUTION HELD BY ELEMENTARY PRESERVICE TEACHERS

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Relatively little research on the distributional thinking has been published, although there has been research on constituent aspects of distributions such as averages and variation. Using these constituent aspects as parts of an conceptual framework, this paper examines how elementary preservice teachers (EPSTs) reason distributionally as they consider graphs of two data sets having identical means but different spreads. Results show that the subjects who reasoned distributionally by considering both averages as well as variability in the data were likelier to see the data sets as fundamentally different despite the identical means used in the task.

INTRODUCTION

The purpose of this paper is to report on research describing the conceptions of distribution held by elementary preservice teachers in response to a task involving a comparison of two data sets. The two data sets were presented in the form of stacked dot plots, and two aspects of statistical reasoning germane to the task were a consideration of the average and variation of the two distributions. While research has uncovered different ways that people think in regards to measures of central tendency (Mokros & Russell, 1995; Watson & Moritz, 2000), fewer studies has been done on how people coordinate averages and variation when comparing distributions. Of particular interest was the role that variation, or variability in data, played in the subjects' conceptions of distribution. This interest stems from the primacy that variation holds within the discipline of statistics (Wild & Pfannkuch; 1999).

Furthermore, although precollege students have been the focus for many researchers interested in statistical reasoning, relatively less attention has been paid to the statistical thinking of the teachers of those students. Even less prevalent has been published research on how preservice teachers reason statistically (Makar & Canada, 2005). Therefore, this study addresses the following research question: What are the informal conceptions of distribution held by EPSTs as they compare two data sets? After describing some related research concerning the aspects of distributional reasoning used in the analytic framework, the methodology for the study will be explicated. Then, results to the research question will be presented, followed by a discussion and implications for future research and teacher training programs.

CONCEPTUAL FRAMEWORK

The key elements comprising the conceptual framework for looking at distributional reasoning are a consideration both aspects of center (average) and variation, which

implies taking an aggregate view of data as opposed to considering individual data elements (Konold & Higgins, 2002). Coordinating these two aspects is what enables a richer picture of a distribution to emerge (Mellissinos, 1999; Shaughnessy, Ciancetta, & Canada, 2004; Makar & Canada, 2005).

For example, Shaughnessy and Pfannkuch (2002) found that using data sets for the Old Faithful geyser to predict wait-times between eruptions provided an excellent context for highlighting the complementary roles of centers and variation in statistical analysis. The question they posed to high school students was about how long one should expect to wait between eruptions of Old Faithful. At first, many students just made an initial prediction based on measures of central tendency (such as the mean or median), which also disregards the variability in the distribution. Shaughnessy and Pfannkuch (2002) point out that “students who attend to the variability in the data are much more likely to predict a range of outcomes or an interval for the wait time for Old Faithful... rather than a single value” (p. 257).

Similarly, Shaughnessy, Ciancetta, Best, and Canada (2004) investigated how middle and secondary school students compared distributions using a task very similar to the task used in the current study reported by this paper. Given two data sets with identical means and medians, subjects showed how they reasoned about averages and variation, with higher levels of distributional reasoning attributed to those responses that conflated both components of centers and spread. The researchers (Shaughnessy et. al., 2004) found that subjects’ conceptions included the notion of “variability as extremes or possible outliers; variability as spread; variability in the heights of the columns in the stacked dot plots; variability in the shape of the dispersion around center; and to a lesser extent, variability as distance or difference from expectation” (p. 29). Their findings and recommendations echoed that of Mellissinos (1999), who stressed that although many educators promote the mean as representative of a distribution “the concept of distribution relies heavily on the notion of variability, or spread” (p. 1).

Thus, recognizing the importance of getting students to attend to both aspects of average as well as variation when investigating distributional reasoning, these two aspects helped inform not only the task creation for this research but also the lens for analysis of the EPSTs’ responses.

METHODOLOGY

The task chosen to look at EPSTs’ thinking about distribution when comparing data sets was called the Train Times task, and was motivated by a similar tasks initially used in previous research (e.g. Shaughnessy et. al., 2004; Canada, 2006). The task scenario describes two trains, the EastBound and WestBound, which run between the cities of Hillsboro and Gresham along parallel tracks. For 15 different days (and at different times of day), data is gathered for how long the trip takes on each of the trains. The times for each of these train trips are rounded to the nearest 5 seconds and are presented in Figure 1.

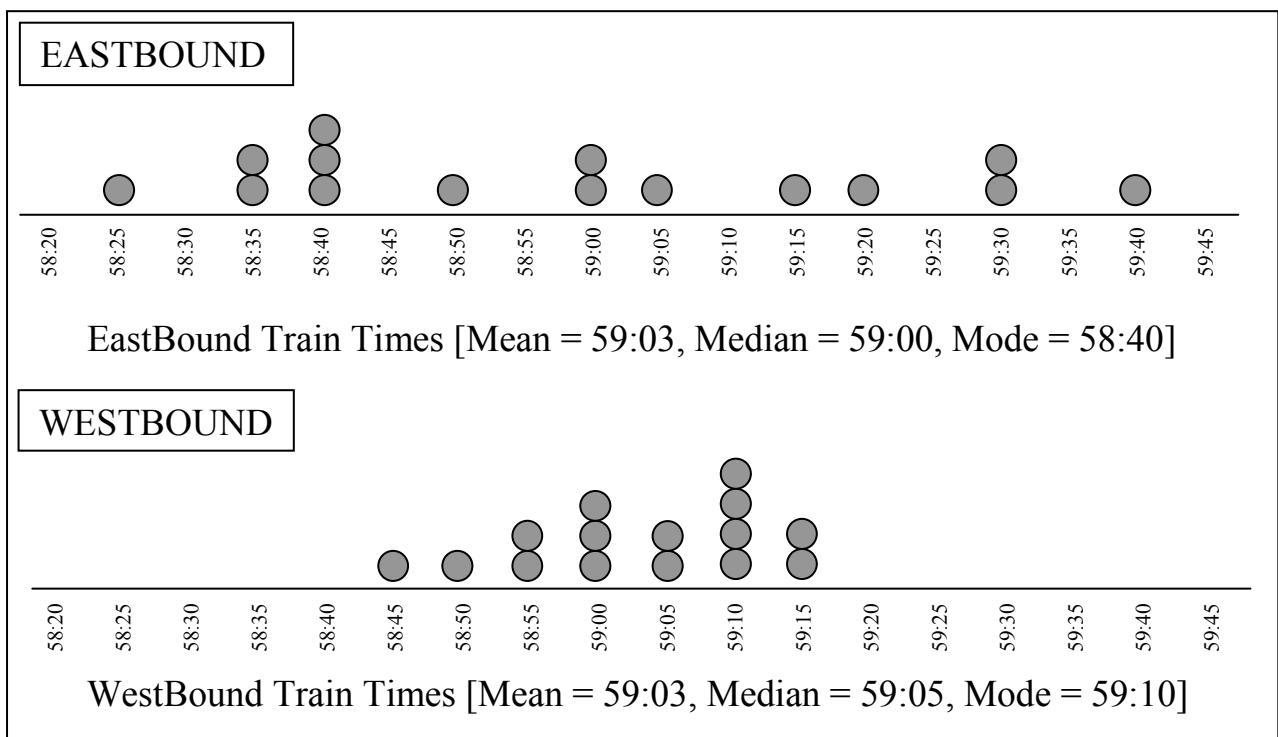


Figure 1: The graphs used in the Train Time task

The task was deliberately constructed so that the EastBound and WestBound train times have the same means, yet different amounts of variation are apparent in the graphs. As a part of the task scenario, subjects were told that the Transportation Department was deliberating whether or not one train was more reliable than the other. Subjects were whether they agreed with a hypothetical argument that there was “no real difference between the two trains because the data have the same means”, and to explain their reasoning. This methodology follows that of Watson (2000), where subjects are asked to react to a common line of reasoning. A similar technique has been used in other research on statistical thinking (e.g. Shaughnessy et. al., 2004; Canada, 2006). Would subjects be persuaded by the hypothetical argument of “no real difference” in times because of the identical means? Would they argue on the strength of the different modes, which are often a visual attractor for statistical novices reasoning about data presented in stacked dot plots? Or would they attend to the variability in the data, and if so, how would they articulate their arguments? The subjects were EPSTs who took a ten-week course at a university in the northwestern United States designed to give prospective teachers a mathematics foundation in geometry and probability and statistics. Virtually none of the EPSTs expressed a direct recall of ever having had any prior formal instruction in probability and statistics, although their earlier education at a precollege level may well have included these topics. Early in the course, and prior to beginning instruction in probability and statistics, subjects were given the Train Times task as a written-response item for completion in class. The task was not given as part of a formal evaluation for the course, but rather as a way of having the subjects show their informal sense of how they were initially thinking. Two sections of the course, taught by the author, were used for

gathering data, and a total of fifty-eight written responses were gathered from the EPSTs. The task was then discussed in class, and the discussions were videotaped so that further student comments could be recorded and transcribed.

The data, comprised of the written responses and transcriptions of the class discussions, was then coded according to the components of conceptual framework which related to the distributional aspects of centers and spread. Responses could be coded according to whether they included references to centers, or to informal notions of variation, or to both.

RESULTS

Almost 35% ($n = 20$) of the EPSTs initially agreed with the hypothetical argument that there was “no real difference between the two trains because the data have the same means” While it might be expected that subjects who were predisposed to think of the mean as the sole or primary summary statistic for a set of data might support the hypothetical argument, a careful analysis of the responses showed different degrees to which subjects relied on centers and variation in their explanations. Thus, in addressing the primary research question (“What are the informal conceptions of distribution held by EPSTs as they compare two data sets?”), results are presented first according to responses that focused primarily on centers, then primarily on variation. Finally, examples of those responses that integrated centers with an informal notion of variation are presented as representing a form of distributional reasoning.

Centers

Out of all subjects, 24.1% ($n = 14$) had responses that included what were coded as *General* references to centers, and the exemplars that follow show the initials of the subject as well as an (A) or (D) to show whether they initially agreed or disagreed with the hypothetical argument of “no real differences” presented in the task scenario:

SE: (A) Because the average is the same for both of them
DW: (A) Each train had the same average time

Although it can be presumed that the subjects equate “average” with the mean, the *General* responses for center included no specific language.

In contrast, the 39.7% of subjects ($n = 23$) who had *Specific* references to centers were more explicit as far as what they were attending to:

SG: (A) The “mean” means the average, so both trains do travel for the same length of time
LT: (D) I would probably go with the mode, because it is the most common answer
RB: (D) I would go by the median on this one

Note how subjects LT and RB, in focusing on the mode or median, disagreed with the hypothetical argument of “no real difference.” Indeed, although the means for the data

sets in Train Times task were identical, the medians and modes differed, and some subjects with *Specific* center responses picked up on these differences:

CM: (D) The median and modes are not the same, meaning results varied
 LN: (D) The median & mode are different. Because the data is very different in its variation

Here we see CM and LN tying their observations of differences in measures of centers to an informal notion of variation.

Variation

For the purposes of this research, variation need not be defined in formal terms such as a standard deviation (for which these subjects had no working knowledge), but tied to the informal descriptions such as those offered by Makar & Canada (2005). In particular, the essence of variation is that there are differences in the observations of the phenomena of interest. Of all subjects, 32.8% ($n = 19$) gave a more *General* reference to variation:

CG: (D) Because the data for both are different in variation of time
 TS: (D) The trains could all have different times sporadically
 LR: (D) Because the time patterns are different between the two trains

Note that in these examples, the theme of differences among data comes out in the natural language of the response. Clearly the sense of variation as differences is a naive and basic idea, but one that is fundamental and a potentially useful springboard for a deeper investigation as to how to describe those differences

A slightly higher percentage of all subjects, 37.9% ($n = 22$) had more *Specific* references to variation, and the main motivation for looking at specific constituent characteristics of variation came from the related literature on thinking about variability in data (e.g. Shaughnessy et. al., 2004; Canada, 2006). These characteristics include relative spread, extreme values, and range. For example, consider these two exemplars of more *Specific* reasoning about variation:

EK: (D) Because looking at the charts, the data is more spread out going EastBound than it is going WestBound
 AD: (D) Because the times for the EastBound trains are very spread out while the WestBound trains' times are clustered together.

Although EK and AD did not capture formal numerical descriptors of variation about a mean, they did use informal language to convey an intuitive sense of the relative spread of data. Other subjects included variability characteristics in their responses by paying attention to extreme values:

AU: (D) No, because the EastBound has more outliers and is more scattered
 AN: (D) One EastBound train took 59:40 while the longest WestBound train took only 59:15, and that is almost a 30 minute difference

Note how AU shows sensitivity to the presence of outliers, while AN includes references to the maximal value in each data set. In addition to *Specific* characteristics

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of variation captured by responses suggesting a focus on relative spread or extreme values, some responses made explicit connection between both maximum and minimum values:

AU: (D) EastBound has a higher range, from 58"25 to 59:40, & WestBound's smaller range is from 58:45 to 59:15

DM: (A) The range is 1:15 seconds EastBound and 0:30 seconds WestBound.

It was interesting to note that even while acknowledging the different ranges, DM still chose to agree with the hypothetical argument in the task.

Distributional Reasoning

As noted in the previous exemplars, some responses focused more on centers and others on informal notions of variation. However, in line with the previous research (e.g. Shaughnessy, Ciancetta, Best, & Canada, 2004), responses coded as distributional needed to reflect an integration of both centers *and* variation. Of the total 58 subjects, 43.1% (n = 25) such responses that reflected distributional reasoning:

HH: (D) Because the mean is an average, and to get an average you will most likely use varying numbers. All the times for the most part on EastBound trains are different. Just because the mean is the same doesn't change that.

AJ: (D) The data is different, although the average is the same. We can see, for example, the difference in consistency of the WestBound train, where the times are closer together, and hold nearer to schedule

AB: (A) The mean is the average of times. BUT, there is a greater spread of times on EastBound vs. WestBound. And East mode is lower than West mode.

Note the richness in the exemplars provided above, as subjects integrate center and spread in their consideration of the two data sets. We see, for example, how HH understands about combining "varying numbers" to get an average. AJ actually lays the groundwork for making an informal inference, in the way that the WestBound may be to more reliable train because it holds "nearer to schedule" Meanwhile, AB agrees with the hypothetical argument of the task, despite apparently taking note of the means and commenting on the differences in mode and spread of the data sets. When asked further for an explanation of his stance in agreeing, he remarked to the effect that "Still, they are basically the same on average". Such is the power of the mean in many peoples' minds as a way of summarizing data.

DISCUSSION

Out of all 58 subjects, 20 (34.5%) initially agreed and 38 (65.5%) disagreed and with the hypothetical argument of "no real differences" between the trains. But this research is about how EPSTs reason distributionally, and so it was crucial to dig into their explanations. The exemplars provided have been intended to show how subject responses could reflect a focus on centers, on variation, or on both. Since responses could be coded for multiple aspects, including some facets of statistical thinking not reported on in this paper, as a final note it is interesting to look at the breakdown of

who agreed versus who disagreed with the hypothetical argument based on whose responses coded *only* for centers, or *only* for variation, or coded for distributional reasoning (both centers and variation). Looking strictly within those groups of responses, the percentage of those subjects who agreed or disagreed is presented in Table 1:

Type of Reasoning		(A)gree	(D)isagree
Only Centers	(n = 12)	75.0 %	25.0 %
Only Variation	(n = 16)	31.3 %	68.8 %
Distributional	(n = 25)	20.0 %	80.0 %

Table 1: Support for “no real differences” by type of reasoning

Again, the percentages in Table 1 are out of the respective numbers of responses falling within the given types of reasoning (the numbers do not total 58 because some students had explanations that went outside the themes of this paper). Comparing those percentages with the total pool of subjects (34.5% agreeing and 65.5% disagreeing), several interesting observations can be made. First, while more than half of all subjects disagreed, the majority of those subjects who only relied on reasoning about centers agreed with the hypothetical argument of “no real differences” Second, the percentage of those subjects who only used variation reasoning and disagreed is quite close to the percentage of all subjects who disagreed. Third and most important, the subjects whose responses reflected distributional reasoning had the highest percentage of disagreement with the hypothetical argument.

CONCLUSION

This study was guided by the question “What are the informal conceptions of distribution held by EPSTs as they compare two data sets?” Although limited to a single task that was by nature contrived so as to invite attention to identical means yet differing amounts of spread in two data sets, the research suggests that EPSTs do reflect on aspects of distributional reasoning, and this paper gives a sense of how those aspects manifested themselves in the responses of the subjects. However, just as prior research has shown with middle and high school students (e.g. Shaughnessy et. al., 2004), we also see that EPSTs make more limited comparisons of distributions when they focus on centers while not attending to the critical component of variability in data. In contrast, the distributional reasoners who attended to both centers *as well as* variation made richer comparisons within the given task.

While further research is recommended to help discern the most effective ways of moving EPSTs toward a deeper understanding of distributions, certainly tasks such as the one profiled in this paper provide good first steps in helping universities offer opportunities to bolster the conceptions of the preservice teachers they aim to prepare.

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In turn, these novice teachers can then promote better statistical reasoning with their own students in the schools where they eventually serve.

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FIRST GRADERS' STRATEGIES FOR NUMERICAL NOTATION, NUMBER READING AND THE NUMBER CONCEPT

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We have come to accept that children's appropriation of written numbers is not automatic or simple. Studies of children's use of notation point to different types of notational practice (Alvarado, 2002; Brizuela, 2004; Scheuer et. al., 2000; Seron & Fayol, 1994). In a recent study (Cayton, 2007) we found that some common strategies of children for representing large numbers were correlated with different levels of understanding of our base-ten system. In this paper, we explore the connections among three representations of the number system, addressing the question: Are there differences in how children represent (through production and interpretation): numbers in writing; numbers orally; and numbers conceptually?

RATIONALE AND PAST RESEARCH

The aim of the study described in this paper was to explore the connections among three representations of the number system in children: nonverbal representations, representations through the written number system, and representations through the oral number system. Past research has usually focused on children's performance on each of these different modes of representation in an isolated way. For instance, previous studies (e.g., Fuson & Kwon, 1992; Miura & Okamoto, 1989; Power & Dal Martello, 1990; Ross, 1986; Scheuer, 1996; Seron & Fayol, 1994; Sinclair & Scheuer, 1993) have focused on place value notation, number decomposition, and oral numeration. Yet there has been very little work on the correlation of any of these aspects with one another.

More recent research has begun to explore the potential connections among these different systems. For example, Scheuer et al (2000) discuss, among others, two distinct types of incorrect numerical notation strategies used by children: logographic (writing the entire number literally, such as 100701 for one hundred seventy-one) and compacted notation (removing some of the zeros from the logographic notation while still not condensing the number entirely into its conventional form, such as 1071 for one hundred seventy-one). Scheuer and her colleagues speculate that perhaps these two types of notational strategies stem from different ideas in children about the concept of number, yet her study only focuses on written numbers. Other studies explore children's use of notation and point to several different types of notational practice possibly linked to stages in the understanding of multi-digit numbers (e.g., Brizuela, 2004) and possibly linked to stages in the understanding of place value in base-ten (e.g., Alvarado, 2002; Brizuela, 2004; Scheuer et al, 2000; Seron & Fayol, 1994). The study we described in this paper is novel in that we seek to explore the connections among these different—yet fundamentally related—modes of representation.

A number of other studies also set the ground for the need to look at children's representation of number across different systems. For instance, Power and Dal Martello (1990) conducted a study of Italian second graders taking numerical dictation of one, two, three, and four digit numerals with and without internal zeros. Since the Italian number words have a similar transparency¹ to other Romance languages such as Spanish as well as to English, findings in this study can be considered relevant to the study here being proposed. The second graders in this study correctly annotated all numbers below 100, demonstrating that they had both succeeded and mastered the number system in this range, making the findings with larger numbers quite remarkable. For numbers above 100, Power and Dal Martello (1990) found both lexical (such as using a 7 instead of an 8) and syntactical (adding extra zeros, improperly arranging numbers, etc.) errors, though the syntactical errors far outweighed the lexical, indicating a misunderstanding of the system and not simply a mistake on the part of the child. There were also significantly more errors for four-digit than three-digit numbers, demonstrating that there are indeed steps in the acquisition of the system and it is not simply an "all or nothing" understanding. That is, each number range poses new problems or elicits prior problems once again. However, one question that arose from Power and Dal Martello's (1990) study was what role the spoken numbers had played in children's performance. For this purpose, Seron and Fayol (1994) devised a follow-up experiment aimed at answering whether the children in the Power and Dal Martello (1990) study comprehended correctly the verbal number forms and to further understand what role language has in this process. In their study, they compared French and Walloon children. French has a very similar numeration system to English and Italian, aside from the forms for 70, 80, and 90 that translate to sixty-ten, four-twenty, and four-twenty-ten respectively. However, in Wallonia, a region of Belgium where French is spoken, the words for 70 and 90 (septante and nonante, respectively) mirror those used in English and most Romance languages. Due to this small difference in the languages, Seron and Fayol (1994) conducted a similar transcoding experiment to Power and Dal Martello (1990) to see if the French and Walloon children would have any differences in numerical dictation or understanding of the written number system caused by the differences in the oral numeration of each language. Seron and Fayol (1994) adopted a longitudinal approach by interviewing children at three sessions, each distant by a three-month interval. In these sessions, children were asked to both transcribe dictated numerals and create the number values from tokens of 1; 10; 100; and 1,000 units. Seron and Fayol (1994) found very similar lexical and syntactical errors to Power and Dal Martello (1990) and Scheuer et al (2000). Not surprisingly, certain incorrect syntactical responses were seen only among French children such as 6018 for "soixante-dix-huit" (seventy-eight) and 42017 for "quatre-vingt dix-sept" (ninety-seven). Interestingly, these errors are very similar to

¹ Transparency/nontransparency (see Alvarado & Ferreiro, 2002) has to do with the degree to which a number sounds like its written form when spoken aloud. For example, in English, 60 is more transparent than 30 since the "six" is clearly heard in the pronunciation of the number. Despite this distinction, transparency is still somewhat relative, because, for instance, we cannot say that 60 is completely transparent since the "ty" does not sound exactly like "ten" or "zero."

those produced by French adult aphasics (DeLoache & Seron, 1982), indicating that this is likely an error at the level of processing or comprehension, not an error at the stage of production (though we cannot be certain, it is always possible that two different sources of error produce the same outcome). The types of tasks at which the children from Wallonia excelled versus erred demonstrate that their difficulties in transcoding from oral numbers to written numbers were mainly due to the production of the written numbers themselves and not a difficulty in understanding what numerosity the spoken number referred to. The same cannot be said for French children, who produced token arrangements very similar to their transcoding errors. As Seron and Fayol (1994) pointed out, it remains to be determined where in the functional architecture of number processing the children's transcoding errors originate. Is it the result of inadequate comprehension of the verbal number forms, difficulties located at the production stage of written numbers, inadequate comprehension of the number system itself, or a conjunction of two or three of those possibilities?

Prior research thus sets the ground for our assumption that connections across representational systems are important, and that fully reflecting and understanding the nature of these connections is a worthwhile endeavor. However, what remains to be understood and answered through the research study we are proposing is: What is the nature of these connections? What is a child's understanding of number and numerosity when they either correctly or incorrectly name a number? When they correctly or incorrectly write a number? When they correctly or incorrectly build a number with tokens or some other tool?

METHOD

Participants

Twenty-seven first grade students (students need to be six years of age by the time they begin first grade) were interviewed individually towards the end of their first grade. The school these children attend is in an urban suburb of Boston, Massachusetts, in the United States of America. The school is ethnically, racially, and socio-economically diverse. In addition, the school provides a two-way bilingual education to children.

Materials and Procedures

Interviews were carried out as clinical interviews (Piaget, 1965). During the course of the interviews, children were presented with the numbers detailed in Table 1. Our goal was to be able to explore children's oral, written, and nonverbal representations of number. Our proposal was to access children's oral representation through their oral naming of numbers; their written representation through their writing of numbers; and their nonverbal representations through their construction, through tokens, of the "value" of the different numbers.

Series	Number Type	1 st	2 nd	3 rd	4 th	5 th
Series 1	Two digit – “transparent” with final 0	40	60	70	80	90
Series 2	Two digit – nontransparent with final 0	50	30	20		
Series 3	Two digit – transparent without 0	43	64	79	88	91
Series 4	Two digit – nontransparent without 0	53	21	19	35	17
Series 5	Three digit – without 0	127	143	324	465	132
Series 6	Three digit – internal 0	101	207	301	401	504
Series 7	Three digit – final 0	300	760	640	430	910
Series 8	Four digit – without 0	1127	3143	4324	5465	7132
Series 9	Four digit – X0XX	3064	2053	1019	4035	5091
Series 10	Four digit – XX0X	2101	3504	1401	4207	1760
Series 11	Four digit – XXX0	1300	3760	2640	1430	1910

Table 1: Numbers presented to children in the different tasks (orally, through different-valued tokens, or in writing)

Children were randomly assigned to one of six task orders:

(1a) OWTO: Numbers were presented orally by the interviewer, from which the child completed the written part of the task. Next, the child completed the token task based on his/her own writing, and finally completed the oral task from the tokens.

(1b) OTWO: Numbers were presented orally by the interviewer, from which the child completed the token task. Next, the child completed the written task based on his/her own tokens, and finally completed the oral task from his/her own writing.

(2a) WOTW: Numbers were presented in written form by the interviewer, from which the child completed the oral task. Next, the child completed the token task based on his/her own oral forms, and finally completed the written task from his/her own tokens.

(2b) WTOW: Numbers were presented in written form by the interviewer, from which the child completed the token task. Next, the child completed the oral task based on his/her own token forms, and finally completed the written task from his/her own oral form.

(3a) TOWT: Numbers were presented in token form by the interviewer, from which the child completed the oral task. Next, the child completed the written task based on his/her oral forms, and finally completed the token task based on his/her own written forms.

(3b) TWOT: Numbers were presented in token form by the interviewer, from which the child completed the written task. Next, the child completed the oral task based on

his/her written forms, and finally completed the token task based on his/her own oral forms.

Each of the tasks has three parts: oral, written, and tokens. Each one of these parts has both a production and interpretation mode: when numbers are presented in tokens, they can be interpreted through writing or through naming orally; when numbers are presented in writing, they can be interpreted through construction of tokens or through naming orally; when numbers are presented orally, they can be interpreted through construction of tokens or through writing.

Oral part of task: In this part of the task, children were asked to read from a piece of paper or from a token composition the numbers in Table 1. Children who correctly name the first two numbers in a series were presented with the next series (see each one of the rows in Table 1). Children who read at least one number incorrectly in a series received 1-3 more numbers until the child's strategy with that category became apparent.

Written part of task: In this part of the task, every child was asked to write at least two numbers from each series. Children who wrote the first two numbers in a series conventionally were presented with the next series. Children who wrote at least one number incorrectly received 1-3 more numbers in that series until the child's strategy with that series becomes apparent.

Tokens part of task: This part of the task was designed for the purpose of understanding the consistencies/inconsistencies in the child's understanding of our number system without the use of notation. In the object numeracy tasks, children were presented with a number of tasks involving the use of tokens of different colors. Tokens were chosen based on the work of Nunes Carraher (1985) performing similar tasks in the understanding of place value in young children and illiterate adults. The child was told that red tokens are worth 1 point, blue tokens are worth 10 points, white tokens are worth 100 points, brown tokens are worth 1,000 points, and maroon tokens are worth 10,000 points. The child was presented with the same numbers as in the previous tasks but in token-form and asked how many points he/she has or was asked to compose a number with the tokens. Once again, children who correctly composed or recognized the first two numbers in a series were presented with the next series. Children who named at least one number incorrectly received 1-3 more numbers until the child's strategy with that series became apparent.

RESULTS

Our first question was: does the manner in which children were first introduced to the numbers (i.e. in writing, orally, through tokens) affect the outcome of their subsequent responses? Table 2 shows the number of incorrect answers given in the first step of the task, according to their testing condition.

Number Series	10-99	100-999	1000-9999	Total
Oral introduction	0/72 (0%)	18/64 (28%)	39/90 (43%)	57/226 (25%)
Written introduction	3/75 (4%)	11/59 (19%)	34/75 (45%)	48/209 (23%)
Token introduction	12/76 (16%)	16/63 (25%)	15/87 (17%)	43/226 (19%)
Total	15/223 (7%)	45/186 (24%)	88/252 (35%)	148/661 (22%)

Table 2: Number of errors with first step of task. Frequencies are shown over total amount of numbers presented in each task condition

As can be seen in Table 2, there were more errors as the number series advanced (that is, more errors in the thousands than in the hundreds and more errors in the hundreds than in the tens). This would be expected as the children have had less previous exposure to the numbers as they increased in length. The Oral and Written modes of presentation had a higher likelihood of resulting in an initial incorrect answer than the Token introduction, as was confirmed with a chi-square test ($\alpha=0.01$). Thus, the format that numbers are presented in does make a difference to children's subsequent production. In our study, children were more likely to make a correct interpretation and re-representation of a number in a different mode when the number was initially presented to them through tokens and they had to subsequently represent the same number either through the written or oral number systems.

Since the children were producing each representation off of their previous one, we wondered if a child who could initially produce an unconventional re-representation of a number in a different mode (e.g., re-represent a number presented through tokens through the written or oral number systems), could later produce a conventional re-representation when asked to use the same mode that the number had initially been presented in, even if this occurred later in the task (e.g., correctly re-represent through tokens a number that earlier had been presented to them through tokens).

	First representation incorrect, final correct				First representation incorrect, final incorrect			
	10-99	100-999	1000-9999	Total	10-99	100-999	1000-9999	Total
Oral Intro.	0/0 (%N/A)	4/18 (22%)	14/39 (36%)	18/57 (32%)	0/0 (%N/A)	14/18 (78%)	25/39 (64%)	39/57 (68%)
Written Intro.	3/3 (100%)	9/11 (82%)	21/34 (62%)	33/48 (69%)	0/3 (0%)	2/11 (18%)	13/34 (38%)	15/48 (31%)
Token	0/12	1/16	1/15	2/43	12/12	15/16	14/15	41/43

Intro.	(0%)	(6%)	(7%)	(5%)	(100%)	(94%)	(93%)	(95%)
Total	3/15 (20%)	14/45 (31%)	36/88 (41%)	53/148 (36%)	12/15 (80%)	31/45 (69%)	52/88 (59%)	95/148 (64%)

Table 3: Ability to recover from initial error among different conditions. Frequencies are shown over total amount of first incorrect representations. All differences in totals are significant ($\alpha=0.01$).

In Table 3 we see that when the numbers were first presented through tokens, the children were the least likely to recover from the error in subsequent productions. Of 43 errors in the first representation when presented with tokens, there were only two instances (5%) of the final representation matching the initial production. The initial oral presentation had a slightly better outcome with 18 of 57 (32%) representations recovering from initial error. Interestingly, when initially presented with a written number, in 33 of 48 cases, (69%) children were able to recover and produce the correct written number in the end.

While it may seem that this is counter to the results described in Table 2, where we found that errors from the written and oral production were more likely than token production, we argue that these findings are complementary: while they have less difficulties with their token productions and interpretations, children are still trying to figure out details of written and oral representations. Some types of initial representations, such as tokens, may be more helpful than others for children to develop their own representations of number. In addition, we find that children are able to remember and produce a correct written number representation despite a lack of ability to produce the number orally or compose its value through tokens.

DISCUSSION

These results elicit some essential questions as to the focus of early mathematics curricula in the United States, where children are typically taught to write numbers through repeated practice and mathematics testing is typically done on paper. It would appear from the above results that children are internalizing the memorization of written numbers without mapping them onto their nonverbal representations. This allows for good recall of written numbers while simultaneously permitting a gap in the bridge between the written and the nonverbal.

In the reverse situation, many children are able to map a nonverbal representation onto a spoken or written number; yet with numbers that they are not yet able to map, they have no method of retrieving the number once the focus is centered on attempting a written and/or verbal form. This could prove to be an obstacle in conceptual problem solving.

The need for further studies examining methods of transfer from one representational mode to another is evident. We intend to follow these same children into the second grade to examine whether the transfer has improved with exposure to larger numbers

and more arithmetical practices which require more than simply retrieval of a written number form. We also intend to analyse the specific types of errors made by children in their different representations, orally, in writing, and through tokens.

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AN EXPLORATORY STUDY OF ELEMENTARY BEGINNING MATHEMATICS TEACHER EFFICACY

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Referring to the significant factors affecting teacher quality, “teacher efficacy” deserves to be in the heart of this dilemmatic evolution. The purpose of this study aimed to examine beginning teachers’ sense of efficacy in elementary schools, as well as its influential factors. Beginning teachers whose background were and were not in mathematics and science were compared to explore the differences of their teacher efficacy. According to research findings, we should devote all efforts to establish a positive and effective learning organization in order to promote their teacher efficacy internally, externally, and promptly starting from the beginning year.

INTRODUCTION

The significance of enhancing teacher quality becomes the core in the process of global educational reform, where teacher preparation programs must take this responsibility (e.g., Holmes Group, 1995; Ministry of Education [MOE], Taiwan, 2001; MOE, Taiwan, 2004; National Research Council [NRC], 2001; Wright et al., 1997; Wu, 2004). The integrity and implementation of the teacher education program had actually a great influence on a teacher’s acquisition of subject matter knowledge and instructional strategy, and even more on teacher efficacy (Chang & Wu, 2006). In another word, teacher efficacy was considered as not only the key indicator on examining the appropriateness and adequacy of a teacher’s personal instructional readiness (e.g. Allinder, 1995; Ashton & Webb, 1986; Denham & Michael, 1981; Rosenholtz, 1989) but also a warning of showing critical problems the teacher education program faced and orienting future directions of its reform movement (Chang, 2003; Chang & Wu, 2006). However, most studies conducted in Taiwan (e.g. Chu-Chen, 2002; Hong, 2002) focused on investigating elementary teacher efficacy “quantitatively” and “generally” (i.e. not specifically for certain subject areas), few of them chose single subject area such as mathematics for their examinations. Consequently, understanding elementary mathematics teacher efficacy under the circumstance of executing practical instruction, the processing trend of their efficacy change, and factors influencing their efficacy qualitatively would be essential and helpful at the current stage. Especially for those beginning teachers who lacked practical teaching experiences, how would they apply theories learned from the pre-service training program to instructional problems they faced on-site? Would the development of their efficacy be influenced while confronting struggles between theories and practices? What would be the trend of their efficacy development and change? These critical issues should be explored qualitatively and deeply.

According to the report of attending “Trends in Mathematics and Science Study 2003 (TIMSS 2003)” from the National Science Council [NSC] (2004), Taiwanese elementary students ranked the fourth position in mathematics. However, their performance had a significantly difference from those at the first (Singapore) and the second (Hong Kong) positions. Further, comparing to the result of TIMSS 1999, there were 16 percentage increases in the response of students who disagreed “I like mathematics”. This finding showed that more and more students join the train of “lacking interests in mathematics”. Why did our students’ achievement and interest step back? What kind of role did teacher efficacy play in affecting students’ achievement and interest?

“Beginning with research in the 1970s (e.g., Armor et al., 1976; Berman et al., 1977), teacher efficacy was first conceptualized as teachers’ general capacity to influence student performance” (Allinder, 1995, p.247). Since then, the concept of teacher’s sense of efficacy has developed continuously and currently is discussed relevant to Albert Bandura’s (1977) theory of self-efficacy, which indicates the significance of teachers’ beliefs in their own capabilities in relation to the effects of student learning and achievement. Ashton (1985) also stated that teacher efficacy, that is, “their belief in their ability to have a positive effect on student learning” (p.142). Several studies further reported, “Teacher efficacy has been identified as a variable accounting for individual differences in teaching effectiveness” (Gibson & Dembo, 1984, p. 569) and had a strong relationship with student learning and achievement (Allinder, 1995; Gibson & Dembo, 1984). Consequently, in order to better understand the reasons of the reduction in mathematical achievement and interest as well as finding out ways of improving both teaching and learning quality, an in-depth processing study would be the best choice at the moment.

PURPOSE AND METHOD

The purpose of this study aimed to examine beginning teachers’ sense of efficacy in elementary schools, as well as its influential factors. Beginning teachers whose background were and were not in mathematics and science were compared to explore the differences of their teacher efficacy. A mixed methods design was employed in this processing study. “Participant Main Survey” and “Mathematics Teaching Efficacy Beliefs Instruments (MTEBI Chinese version, Chang, 2003)” were used as the instruments of the quantitative part of this research. Participants were 64 beginning elementary mathematics teachers in Taichung, Taiwan. Pre- and post-tests were administered to obtain their efficacy ratings quantitatively. In the qualitative part, beginning teachers with and without background in mathematics and science were selected purposefully as participants according to their efficacy ratings of pre-tests. They were then divided into three groups: high, medium, and low; three teachers were randomly selected from each group. All together, six beginning mathematics teachers participated in the qualitative part of this study. Influential factors to beginning teachers’ sense of efficacy were identified through interviews, recordings, observations, and researchers’ reflection notes for exploring practical strategies to

improve their efficacy. The analysis in context strategy was employed for reaching the research objectives, which was the integration of descriptive and inferential statistical analyses (i.e. ANCOVA) and the qualitative analysis.

FINDING AND DISCUSSION

Quantitative teacher efficacy comparison

Findings of this study were reported in two parts. For the quantitative part, there were 18 elementary beginning mathematics teachers with mathematics and science (M&S) background and 46 without mathematics and science background. In regard to the pre-test, beginning mathematics teachers with M&S background had a significantly superior rating in both cognitive dimensions of Personal Mathematics Teaching Efficacy (PMTE) and Personal Science Teaching Efficacy (PSTE). After receiving four-years of training, beginning mathematics teachers with M&S had more confidence in their own teaching abilities than those who did not specialize in either mathematics or science. With regard to the post-test (one year later), beginning mathematics teachers with M&S background still scored significantly higher in both PMTE and PSTE than those without M&S background. In order to obtain a better understanding of the differences between these two groups (i.e. with and without M&S background) in the performance of PMTE and MTOE, a one-way analysis of co-variance (ANCOVA) was conducted to determine the effects of the two backgrounds on the efficacy scores. After eliminating the differences in the pre-test, beginning mathematics teachers with M&S background still scored significantly superior in both PMTE and PSTE in the post-test than those without M&S background. Table 1 showed the comparative statistics of PMTE and MTOE.

Program		Value	Sig.	Mean Differences ¹
Pre-test	PMTE	$t=2.808$	$p<.01$	3.379
	MTOE	$t=3.393$	$p<.001$	2.297
Post-test	PMTE	$t=4.947$	$p<.001$	5.461
	MTOE	$t=4.958$	$p<.001$	4.292
ANCOVA	PMTE	$F=19.770$	$p<.001$	
	MTOE	$F=18.759$	$p<.001$	

Note: 1. Mean Differences = Mean with M&S – Mean without M&S

Table 1: Comparative Statistics of PMTE and MTOE

However, considering the average mean scores of PMTE and MTOE, those beginning mathematics teachers without M&S background only had approximately 71.94 percent of confidence (post-test) in their own teaching abilities. This information provided a warning for all teacher training programs: If these beginning teachers believed they were not ready to assume the teaching responsibility, teaching quality would be potentially jeopardized. Moreover, they did not have adequate confidence (only

72.83%) in providing efficient teaching in the classroom either. Thus, even though they believed effective teaching was vital for students' learning and achievement in mathematics, the quality of teaching and learning could still not be assured. These findings just matched the results of previous studies, such as Ball (1990), Chu-Chen (2002), Hong (2002), and Chang (2003). As Gibson and Dembo (1984) stated, teachers with high efficacy should "persist longer, provide a greater academic focus in the classroom, and exhibit different types of feedback than teachers' who have lower expectations concerning their ability to influence student learning" (p.570). Further, "when it comes to the education of our children...failure is not an option!" said President George W. Bush (2001). Accordingly, since there was no time for waiting and no room for going back and regret, how to enhance beginning mathematics teachers' efficacy quickly and effectively for the purpose of providing quality teaching process and learning environment would be urgent and critical task for all in-service training programs.

Influential Factors of Teacher Efficacy

For the qualitative part, 6 beginning mathematics teachers were selected for discovering influential factors of their teacher efficacy. Table 2 and 3 showed their background information. According to the qualitative findings, two categories of factors that influenced the change of their teacher efficacy were generalized, teacher's teaching belief and practical instruction (internal factor) and peer interaction and administrative support (external factor). The internal factor had three sub-categories: mathematics background knowledge and previous experience, instructional belief and action, and teacher-student interaction. The external factor was divided into three sub-categories too: peer interaction, administrative support, and teaching resource.

Teacher Efficacy	Sex	Grade Level	Major	Pre-test(total score) (average: 80.61)
M1	High	M	6	Science Education
M2	Middle	M	3	Math Education
M3	Low	M	3	Math Education

Table 2: Information of Teachers with M&S Background

Teacher Efficacy	Sex	Grade Level	Major	Pre-test(total score) (average: 74.93)
N1	High	F	5	Food Science
N2	Middle	F	5	Music Education
N3	Low	F	4	Art Education

Table 3: Information of Teachers without M&S Background

A. Internal Factor

First of all, the findings indicated that beginning mathematics teachers had inadequate mathematical background knowledge and practical experience before they entered the

classroom. This inadequacy led to several obstacles, such as the difficulty in preparing the lessons, mistakes made in the teaching process, and low teacher efficacy. For example, when students proposed a difficult question, teachers with low efficacy usually could not decide how to deal with this situation. Same conditions happened while students applied various strategies to solve the problem that was different from the strategy used by the teacher. What should the teacher do? Should s/he explain and present every strategy that could solve the problem or just ignore it? This dilemma would definitely resulted in a inadequate and ineffective instruction and learning. As Allinder (1995) indicated, "Teachers with high personal efficacy and high teaching efficacy increased end-of-year goals more often for their students ... Teachers with high personal efficacy effected significantly greater growth" (p. 247). While thinking how to promote mathematical teaching outcomes, educators should consider deeply how to increase beginning mathematics teacher efficacy first.

Secondly, beginning mathematics teachers who had low efficacy tended to have insufficient instructional strategy and bad teacher-student interaction. They usually did not know how to propose questions and guide the classroom discussion. Consequently, they mostly used "lecture" while teaching. Even if they had a discussion, it was always ineffective. This situation led to not only decrease their teacher efficacy but also reduce students' learning interests and motivations, as well as less teacher-student interaction. This finding corresponded with Czerniak's (1990) opinion: Teachers with a high sense of efficacy have been found to be more likely to apply inquiry and student-centered teaching strategies, while low efficacious teachers are more likely to use teacher-centered strategies, e.g. lecture and reading from the textbook. Therefore, teacher efficacy did play a significant role in considering how to improve the quality of teaching, and was one reasonably important part of learning quality. Only if the teacher efficacy were increased, students' interests in learning mathematics and their active learning habits would be promoted effectively.

Another internal sub-category was techniques applied for classroom management. In this study, beginning mathematics teachers with low efficacy were likely to spend a great deal of time in managing the classroom order. They often felt powerless and pressured, which led to an unsuccessful teaching. In fact, employing body language and specific movements appropriately could keep students' attention to the instruction and cultivate positive agreements between the teacher and students simultaneously. Thus, providing relative information of classroom management would complement the instructional strategies and teacher-student interactions effectively for enhancing both teaching and learning quality.

B. External Factor

All six beginning teachers mentioned that they felt more comfortable and confident in preparing or implementing their instruction once receiving active care and assistance from experienced teachers or teachers of the same grade level within the school. Several studies (e.g. Piaget, 1970; Rogoff, 1990; Saxe, Gearnart, & Guberman, 1987) indicated that these experience sharing and encouragements were so helpful that they

learned a lot and felt full of enthusiasm to keep going forward. However, they all complained of the inadequacy of the sharing and encouragement environment. Consequently, how to manage the working environment within one elementary school, build up the opportunity of cooperative learning among all teachers, and promote the interchange of instructional knowledge and experience would be crucial to assist beginning mathematics teachers' professional development. Further, the school administrators should rethink their management belief and strategy for the purpose of establish an effective learning organization for all teachers (Borko, 2004), where they could learn from each other efficiently and grow together professionally.

Moreover, all six teachers indicated that the school administrative level excessively interfere their teaching in the classroom. This interference worsened the communication channel and led to an unfavourable relationship that became another external factor of the decrease of teacher efficacy. In addition, beginning teachers were often assigned to participate in competitions within or outside of the school that were not relevant to their major or the subject they taught. They had to spend more time for these extra tasks and resulted in less time for preparing their own teaching in the classroom. Consequently, instead of giving extra interferences and duties, the school administrative level should reconsider what realistic supports should be provided for beginning teachers in order to assist them to enhance their teacher efficacy and focus on their own teaching.

CONCLUSION

According to the findings of this study, beginning mathematics teachers who majored in mathematics and science education had a significantly higher increase in their efficacy ratings than those who did not major in mathematics and science both at the beginning and the end of the first year. Also, beginning mathematics teachers who majored in mathematics and science education had a significantly higher increase in both personal teaching efficacy and teaching outcome expectancy than those who did not major in mathematics and science. Further, two categories of factors found in this study influencing beginning mathematics teacher efficacy included: teacher's teaching belief and practical instruction (internal factor) and peer interaction and administrative support (external factor). As Hermanowicz (1966) and Ladd (1966) stated, "Teachers repeatedly have indicated that their teacher training did not prepare them to be effective teachers. Many have made suggestions for improving teacher education" (p. 53). Benz et al. (1992) further confirmed their opinion 26 years later. Especially for those mathematics beginning teachers, under the condition of having low teacher efficacy and inadequate readiness in teaching, how to assist them in regard to their belief, confidence, and practical instruction would be the first task of teacher professional development. Accordingly, as teacher educators, we should reflect from these previous recommendations and research findings, and further devote all efforts to establish a positive and effective learning organization in order to promote their teacher efficacy promptly starting from the beginning year.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE AND INQUIRY TEACHING APPROACHES

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This paper discusses the nature and role of preservice secondary mathematics teachers' knowledge that supports their use of inquiry approaches during their practicum teaching. It highlights how four categories of teachers' knowledge and, more importantly, the connectedness among them based on a common theme influenced the preservice teachers' use of inquiry approaches and their ability to transform pedagogical theory to practice. The paper also addresses the importance of learning experiences in teacher education that treat these domains of teacher knowledge in an integrated way within classroom-based contextual situations in order to facilitate the development of an appropriate, usable network of knowledge.

This study investigated the nature and role of preservice secondary mathematics teachers' knowledge that supported their use of inquiry approaches during their practicum teaching. It is part of an ongoing four-year longitudinal study of beginning secondary mathematics teachers' growth.

RELATED LITERATURE

An inquiry perspective of teaching is considered to be effective to facilitate students' development of mathematics understanding and mathematics thinking. In inquiry classrooms, students are expected to construct mathematical meaning through reasoning, communicating, exploring, and collaborating with peers and the teacher while working on tasks that are inquiry-oriented activities, including genuine problems and investigations (NCTM, 1991). However, teachers typically find it challenging to adopt inquiry approaches in their teaching, particularly at the secondary level, partly because to teach this way requires teaching differently from how they were taught. Current teacher preparation programs are likely to expose prospective teachers to such approaches to various degrees. However, based on my experience with an inquiry-based program, this exposure, regardless of how substantive, may not result in implementation in the classroom during practicum, even when the supervising teachers give the student teachers the freedom to do so. This raises the issue of when and how are prospective teachers able, or likely, to implement inquiry approaches in their teaching during practicum and potentially in their future practice. This study contributes to our understanding of this issue.

The research literature provides evidence to support concerns about the adequacy of preservice teachers' knowledge as a basis to teach mathematics in an inquiry way. For example, research studies that examined preservice secondary mathematics teachers' mathematics knowledge (e.g., Even, 1993; Feiman-Nemser & Remillard, 1996;

Kinach, 2002; Knuth, 2002) suggest that these teachers often do not hold a sound understanding of the mathematics they need in order to teach it with depth. This includes fundamental concepts from the school curriculum, such as operations with integers, functions, and proof. In contrast to knowledge of mathematics, studies on preservice teachers' knowledge of mathematics teaching and learning are less represented in the research literature (Ponte & Chapman, 2006). However, some studies show that they could have inadequate understanding of students' mathematics thinking (Tirosh, 2000; Stacey et al., 2001) and knowledge of communication and questioning (Moyer & Milewicz, 2002; Blanton, Westbrook, & Carter, 2005).

In general, studies on preservice teachers' knowledge tends to focus on what they do not know than on their sense making (Ponte & Chapman, 2006), including the nature of their knowledge that supports inquiry-oriented teaching. In focusing on their sense making and practical knowledge, this study is framed in a humanistic perspective of teacher thinking in which teachers are viewed as creating their own meaning to make sense of their teaching (Fenstermacher, 1994). It also adopts the perspective of Kilpatrick, Swafford and Findell (2001) that three major components of mathematics teacher knowledge are necessary for effective mathematics teaching: knowledge of mathematics, students, and instructional practices.

RESEARCH PROCESS

The study was framed in a qualitative, naturalistic research perspective that focuses on capturing and interpreting peoples' thinking and actions based on actual settings. Case studies (Stake, 1995) were conducted to allow for in depth examination of the situation. The participants were two secondary preservice mathematics teachers with bachelor degrees in mathematics and were in the second year of their two-year B.Ed. program, which had a focus on inquiry teaching. The study was built around their semester-3 practicum when they were in their assigned schools teaching for most of the semester, four days per week. During this semester, they integrated inquiry approaches in their teaching in spite of being in predominantly traditional classrooms.

The main sources of data were interviews, classroom observations and teaching documents. The open-ended interviews before and after the semester included a focus on: their thinking about mathematics and inquiry teaching and learning; their meanings/interpretations of selected mathematics concepts covered in their teaching; their actual experiences with inquiry approaches during their practicum teaching; their thinking behind planning and conducting their lessons; and their thinking about what supported or inhibited their use of inquiry approaches. Classroom observations were conducted for lessons including and not including inquiry approaches. Data from the observations included teacher-student interactions about the mathematics topics being taught; description of learning activities students engaged in; what the teacher attended to as students worked on mathematics tasks; and teaching strategies for the content. Documents consisted of lesson plans, teacher prepared mathematics activities and field journal records of their reflections on their teaching.

Analysis involved close scrutiny of the data by the researcher and research assistants, focusing on identifying the participants' knowledge and thinking about mathematics, students' learning and inquiry teaching; explicit and implicit situations of when, how and why they used inquiry approaches; and apparent relationships between their knowledge or thinking and inquiry teaching. Themes were determined by identifying conceptual factors that characterized each participant's thinking and practice based on the information from the initial scrutiny of the data. "Pattern" emerged as the most dominant theme in relation to inquiry teaching, then, the data were scrutinized further in order to understand the nature of this theme. Verification procedures included triangulation by comparing outcome from the various data sources, cross checks by research team, member checks and elimination of initial assumptions/themes based on disconfirming evidence.

FINDINGS

The two preservice teachers differed in how they planned and conducted their lessons during their practicum teaching, but they exhibited some key similarities that seemed to characterize their sense making of using inquiry approaches. These similarities involved four factors: their beliefs about mathematics, their beliefs about students' learning, how they held their knowledge of mathematics and their pedagogical knowledge of engaging students in inquiry. In particular, the relationships among these factors seemed to be the key to explaining their use of inquiry approaches. The following highlights the nature of these factors and the relationships among them.

Beliefs about mathematics: The participants held a similar core belief about mathematics that provided the foundation for when and how they used inquiry approaches. For Sara, this belief was "patterns, making connections between patterns and the world," for Reba, "a lot of patterns ... can be found everywhere." They also held the belief in a way that was "central and psychologically strong" (Green, 1971).

Beliefs about students' learning: The participants also held a similar core belief about students' learning that was compatible with inquiry learning and directly related to the belief about mathematics. This belief focused on having students "make the connections for themselves" (Sara) or "see patterns for themselves" (Reba). As Sara explained, "I always say, can you see a pattern? Like I always said, can you see it? Look for it." She added, "It was funny because by the end of the class some of the kids would pick up on that and ask, are we looking for patterns again? Yes, we're always looking for patterns." The belief also focused on "allowing students to develop their own ideas about things ... to find their way to the answer ... giving them the space to be able to do that." (Sara) Similarly, "not that they are discovering everything on their own, but using what knowledge they have, ... [then] to be able to make the connections themselves, with each other and being able to ask themselves questions like: does this work? Does this make sense?" (Reba)

Mathematics knowledge: The way the participants held their knowledge of mathematics played a significant role in terms of when and how they were able to

transform their beliefs about mathematics and learning into practice. For example, mathematics concepts or procedures that they readily understood, or already held, in terms of patterns were taught through inquiry-oriented approaches. This seemed to be more important than whether they had conceptual versus procedural understanding, since deep understanding of the mathematics was not always present or demonstrated with use of the inquiry approach. The participants were able to associate the concepts and procedures they held as patterns with specific inquiry-learning approaches that focused on the mathematical structure of the concepts or procedures. These approaches involved using a compare/contrast technique with tasks of the form of examples versus non-examples; alternative representation of the same concept; card sorting; concrete situations; and solved examples. Thus, the preservice teachers were able to select or construct tasks that embodied the beliefs about mathematics and learning when they (mathematics, concepts, learning) were all viewed as directly related to pattern as structure or way of knowing. The following are two examples of these mathematical/pedagogical tasks that they created.

Both examples consist of concepts or procedures that were new for the students. Sara's example: This task dealt with prime and composite numbers in her Grade 7 class. Sara wrote the numbers 1 through 20 on the blackboard. Students were required to work in groups to find the factors of each number based on their prior knowledge. Then, they "must look for patterns found in the factors" and be prepared to share and discuss them in whole-class setting.

Reba's example: This task dealt with solving systems of linear equation through elimination and substitution as required by the curriculum for her Grade 11 class. Reba prepared three solved examples of elimination and three of substitution. The following is the set of equations used for the elimination cases. A different set was used for substitution.

- (i) $3x - 5y = -9$; $4x + 5y = 23$
- (ii) $2x + 3y - 32 = 0$; $3x - 2y - 22 = 0$
- (iii) $\frac{x-3}{2} - \frac{y-5}{3} = 1$; $\frac{x+3}{2} + \frac{y-3}{4} = 1$

Students were required to "figure out how each process works through the solved examples ... [and] to be able to explain fully how and why each method works ... how do the examples differ and compare with one another." [Sara]

The participants' beliefs about mathematics included connections to the world. But this did not play a significant role in when or how they intentionally used inquiry approaches in teaching a mathematics concept. This aspect of their belief was reflected only in the use of word problems where they tried to include real world-like situations or something to make the task fun in a problem-solving context. Sara did this more often than Reba because the junior high school context she was in seemed to lend itself better to that than the senior high context for Reba. In Sara's case, a few of the problem solving tasks she used allowed for non-algorithmic skills to arrive at a solution. For

example, she explained, “My key is to make them [the tasks] engaging. ... When we did word problems ... I had these plastic eggs and I stuck things inside them. ... They had to find out what was inside the egg without opening it, and they had to talk about it.” Both participants also, on a few occasions, on introducing a topic provided some historical information about it.

Instructional knowledge: Finally, the participants held instructional knowledge for engaging students in the learning tasks that was directly related to their beliefs about learning. In particular, their thinking and practice indicated that this knowledge included use of groups, open/probing questioning, and flexible listening in an inquiry -learning context. For example, Sara explained: “I always tried to make them tell me what they were doing. They would ask a question and I would always try to re-ask the question to them.” She later expanded on this. “[I] say, what do you mean by that? Or where are you going? Or what are you doing with that? So I always tried to listen to their process ... how are they thinking and why.” Also, as Reba noted: “I would ask them well what are you thinking of because then that could maybe trigger them without me having to say to them how do you exactly solve this.”

Practice: Classroom observations of the preservice teachers’ practice revealed that their inquiry-oriented lessons had a similar structure. Each participant was unique in how this structure was lived in the classroom, but an example from Reba’s case will be used to illustrate the structure, which consisted of the following four stages.

An introduction stage: This differed based on the topic and included a check of students’ prior knowledge; brief history of the mathematics concept; and clarifying tasks. For Reba’s Grade 10 introductory lesson on coordinate geometry involving length, midpoint and slope of line segments, she introduced the lesson through a story explaining the history of Descartes and how the coordinate plane was invented. She also clarified the task and explained the unfamiliar notion used for a point.

An exploration stage: This involved students’ working on tasks in groups with the teacher posing questions and prompts. For this stage, Reba prepared cards consisting of different representations of the concepts. One set of cards had “length of line segment”, “mid-point of line segment”, and “slope of line segment.” The other sets of cards contained, for each of the three concepts, equations, graphs, problems, and solutions of the problems. Students were required to sort the cards according to the three concepts and be able to explain why their solutions made sense – “how do you know that these go together?” Students first worked in pairs then groups of four to compare and discuss their findings. Reba, in response to their questions, or her observations, would prompt them to think about what they were doing or not noticing.

A sharing and discussion stage: Here, students shared and defended their findings in a whole-class setting. Reba guided the discussion to make sure they covered all she expected them to know about the concepts.

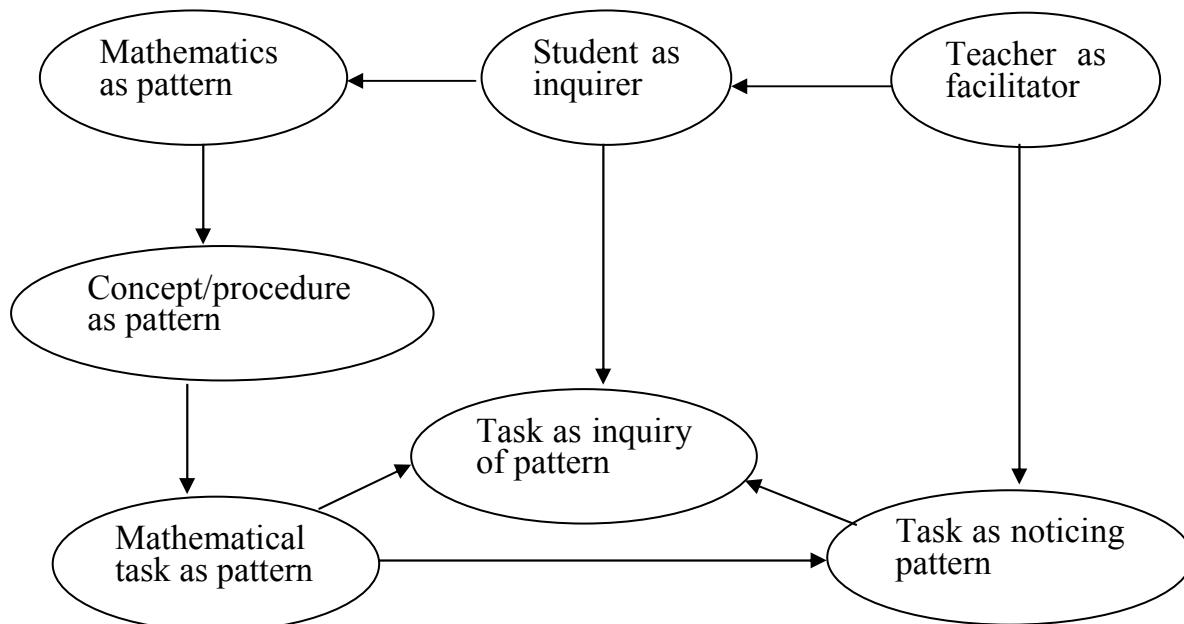
A conclusion stage: Here, Reba guided the students to think about what they learnt about the concepts and summarized the key ideas.

This brief outline of the lesson does not include the ongoing teacher-student interactions that occurred to facilitate the students' thinking, inquiry, and sharing processes that were necessary for the lesson to have an inquiry tone.

DISCUSSION

The four categories of the preservice teachers' knowledge described above are key factors in accounting for, and understanding, their use of inquiry approaches in their teaching. But this importance of the four categories of knowledge lies in their interconnectedness. This connectedness provided the two preservice teachers with a logical and viable image of what this teaching could look like. It was necessary for these teachers to transcend other factors, like contextual constraints, in order to teach with inquiry-oriented techniques. In fact, when the interconnectedness was lacking, for example, the mathematics concept was viewed as a fact instead of as a pattern, these teachers resorted to traditional teaching and justified it in terms of the contextual constraints they perceived, for example, teacher-centered classrooms; limited time; pre-established classroom tone; pre-established sequence of units; nature of topic. As Sara noted: "As a student [teacher], you are living in the partner teachers environment, so this puts constraint on how open-ended I can be in my teaching."

This integrated view of the preservice teachers' knowledge could be considered as a network of the preservice teachers' understandings of key ideas of mathematics, students and instruction and, more importantly, of relationships among them. The following model offers a way of conceptualizing the relationships as held by the preservice teachers. The model connects the preservice teachers' beliefs and



conceptions of mathematics, mathematics concepts and procedures, students and instructional practices when they support inquiry-oriented teaching approaches. In this model, pattern, as it relates to mathematical structure, is the main organizing theme

that connects mathematics, learner and inquiry pedagogy as follows. First, mathematics as pattern is associated with a mathematics concept/procedure that is viewed in terms of its mathematical structure and, thus, as pattern. Then, this concept or procedure is associated with a mathematical task designed in terms of this pattern. Second, the student is viewed in the role as inquirer in order to learn to think about and see mathematics as pattern. The mathematical task becomes a learning task for the student that involves inquiry of pattern. This learning task is framed in a social context. Third, the teacher is viewed in the role as facilitator in order to support students as inquirers. The mathematical task becomes an instructional task for the teacher that involves using prompts and questioning to help students to notice relevant patterns built into the task.

This model indicates that the preservice teachers constructed an image of inquiry-oriented practice that involved understanding mathematics as patterns, understanding learner as inquirer of patterns and understanding teacher as facilitator of student as inquirer of patterns. As can be expected of novice teachers, these understandings, and thus the image of practice, lacked depth. But, in spite of the quality of their knowledge, whether these understandings were held in a disconnected way or connected way that made sense to them was of more importance in shaping their practice. This connection was important to provide the preservice teachers with an intention of teaching that was explicit and concrete, that is, to help students to see “pattern” as a way of learning and understanding mathematics.

This connectedness, then, was necessary for the preservice teachers to make sense of how to transform their newly constructed theoretical pedagogical knowledge into practical knowledge. In this case, when this connectedness was missing, usually because of lack of association between mathematics as pattern and concept and tasks as pattern, although the teachers still had knowledge about inquiry-oriented pedagogy, there was little evidence of it in their teaching as they resorted to more traditional, teacher-oriented practice. Thus, for preservice teachers such as these participants, the key to developing the connectedness is understanding mathematics as pattern conceptually and pedagogically. This is important for them to be able to select or create mathematical tasks that are relevant and meaningful for investigating patterns and to understand the ways of thinking and roles of the learner and teacher in the inquiry. Thus, to add depth to their practice, they need experiences that will deepen this understanding of mathematics as pattern conceptually and pedagogically. In the context of teacher education, this requires learning experiences that integrate and allow for the integration of knowledge of mathematics, learner and instruction.

CONCLUSION

The findings suggest that preservice secondary teachers could transform theory to practice regarding inquiry teaching if they construct a relevant, integrated view of the three core domains of teacher knowledge – mathematics, instruction and students. This allows them to organize their thinking coherently about what and how they intend to

teach. The findings also suggest that it is important to provide them with experiences that treat these domains of teacher knowledge in an integrated way. Rather than treating the knowledge separately, as is often done in teacher education, an approach that treats them as interwoven within classroom-based situations could help prospective teachers to develop knowledge that is useful and usable.

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DEVELOPING AND TESTING A SCALE FOR MEASURING STUDENTS' UNDERSTANDING OF FRACTIONS

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The study reported in this paper is an attempt to develop and test a scale for measuring students' understanding of fractions. In developing the scale, several criteria proposed in previous studies for examining students' construction of the notion of fractions were employed. A test consisting of 44 tasks related to the five subconstructs of fractions was developed and administered to 351 fifth graders and 321 sixth graders. A two-parameter Item Response Model was used and the scale developed was analyzed for reliability and fit to the data. The analysis of the data revealed that the scale had satisfactory psychometric properties. Hierarchical cluster analysis also suggested that the 44 tasks within the scale could be grouped into three clusters, according to their level of difficulty. The findings of the study are discussed with respect to teaching and learning fractions.

INTRODUCTION

In the mid 1970s Kieren (1976) proposed that the concept of fractions is multifaceted and that it consists of five interrelated subconstructs: part-whole, ratio, operator, quotient and measure. Since then, several researchers have put forth and tested a number of criteria for examining students' understanding of the different subconstructs of fractions (e.g., Baturo, 2004; Boulet, 1999; Lamon, 1999; Marshall, 1993; Stafylidou & Vosniadou, 2004). Yet, the extent to which those criteria can form a scale for measuring students' understanding of fractions remains an open question. The present paper aims to address this research gap by developing and testing such a scale. The summary of the relevant literature that follows provided guidelines for the development of a test assessing students' construction of the concept of fractions.

The part-whole subconstruct of fractions is defined as a situation in which a continuous quantity or a set of discrete objects are partitioned into parts of equal size (Lamon, 1999). To develop the part-whole subconstruct of fractions, students should understand that the parts into which the whole is partitioned must be of equal size; they should also be able to partition a continuous area or a discrete set into equal parts and discern whether the whole has been partitioned into equal parts. In addition, they should develop the idea of inclusion or embeddedness (i.e., the parts of the numerator are also components of the denominator) and understand that as the number of parts into which the whole is divided increases, their size decreases (Boulet, 1999). Finally, a full understanding of the part-whole subconstruct requires that students develop unitizing and reunite abilities (Baturo, 2004) which allow them to reconstruct the whole based on its parts and repartition already equipartitioned wholes.

The ratio subconstruct considers fractions as a comparison between two quantities. To grasp the notion of fractions as ratios, students need to construct the idea of relative amounts (Lamon, 1999). They should also comprehend the covariance-invariance

property, according to which the two quantities in the ratio relationship change together so that the relationship between them remains invariant. The “orange juice task” (Noelting, 1980), in which children are asked to specify which mixture of orange juice makes the juice the most “orangey” has been widely employed to examine whether students have developed these ideas.

In the operator interpretation of fractions, rational numbers are regarded as functions applied to some number, object, or set (Behr et al., 1993; Marshall, 1993). One could think of the operator as an application of the numerator of the fraction to the given quantity, followed by the denominator quantity applied to this result, or vice-versa. Alternatively, one could consider the operator as a transformer that changes the size – but not the shape – of a figure or changes the number of elements in a set of discrete objects (Lamon, 1999). To master this subconstruct, students should be able to identify a single fraction to describe a composite multiplicative operation (i.e., a multiplication and a division), and relate inputs and outputs (e.g., a $\frac{3}{4}$ operator results in transforming an input quantity of 4 into 3) (Behr et al., 1993).

According to the quotient subconstruct, fractions are the result of division, in which the numerator defines the quantity to be shared and the denominator defines the partitions of the quantity. To develop an understanding of this subconstruct, students need to be able to relate fractions to division and understand the role of the dividend and the divisor in this operation. Mastering the quotient subconstruct also requires that students develop a sound understanding of partitive and quotitive division (Marshall, 1993).

The measure subconstruct conveys the idea that a fraction is a number; this subconstruct is also associated with the measurement of the distance of a certain point on a number line (Marshall, 1993; Stafylidou & Vosniadou, 2004). Several researchers have argued that, despite their relative understanding of the aforementioned subconstructs of fractions, many students appear not to fully understand that fractions are an extension of the number system (Amato, 2005; Hannula, 2003). Hence, Lamon (1999) refers to a qualitative leap that students need to undertake when moving from whole to fractional numbers. A robust understanding of the measure notion requires that students comprehend that between any two fractions there is an infinite number of fractions. They should also be capable of locating a fractional number on a number line and identify a fractional number represented by a point on a number line (Hannula, 2003).

In this context, the present study sought answers to two research questions. First, to what extent can the criteria proposed above help develop a scale with good psychometric properties to measure students’ understanding of fractions? And second, provided that such a scale can be developed, do tasks that measure different subconstructs of fractions differ in their level of difficulty and their contribution to the development of the scale? Answers to these questions are important both for teaching and assessing fractions.

THE DEVELOPMENT OF THE TEST

To address the research questions, a test on fractions (available on request) was developed taking into consideration the preceding literature review and the curriculum used in Cyprus where the study was conducted. Table 1 presents the specification table that guided the construction of the test and the tasks employed to examine students' performance on each of the five subconstructs of fractions.

Subconstruct	Tasks
<i>Part-whole</i>	1-6, 20, 22-28, 39
<i>Ratio</i>	9, 10, 13-15, 25, 30, 31, 38
<i>Operator</i>	8, 11, 12, 18, 44
<i>Quotient</i>	7, 29, 40, 41, 43
<i>Measure</i>	16, 17, 19, 21, 32-37, 42

Table 1: Specification table of the test used in the study

The first six tasks of the part-whole subconstruct asked students to identify the fractions depicted in discrete or continuous representations. Tasks 22, 27, and 39 examined students' unitizing and reuniting abilities (Battoro, 2004), whereas the remaining tasks addressed common misconceptions related to the notion of embeddedness, the requirement that the parts be of equal size and the inverse proportional relationship of the size and the number of parts into which a unit is partitioned (Boulet, 1999). Nine tasks were used to examine the development of the notion of fractions as ratios. Those included expressing the relative size of two quantities using a fraction (tasks 9-10), identifying fractions as ratios (task 25) and comparing ratios, based either on quantitative (13-15) or qualitative information (30-31) (the latter five tasks were related to the "orange juice problem"). Task 38 referred to the widely cited problem of boys and girls sharing different numbers of pizzas (Marshall, 1993). The tasks used to examine the operator notion of fractions asked students to specify the output quantity of an operator machine given the input quantity and a fraction operator (tasks 11-12). The remaining three tasks required students to decide the factor by which the number 9 should be multiplied to become equal to 15 (task 8), use a fraction to describe a composite operation (task 18), and specify the factor by which a picture reduced by $3/4$ should be enlarged to restore it to its original size (task 44 Lamon, 1999). Three of the tasks used to measure the quotient subconstruct (tasks 29, 40, and 41) examined students' ability to link a fraction to the division of two numbers and identify the role of the dividend and the divisor; the remaining two tasks of this category were related to the partitive and quotitive interpretation of division (tasks 7 and 43, respectively). Consistent with previous studies (Hannula, 2003; Lamon, 1999; Stafylidou & Vosniadou, 2004), the tasks of the measure subconstruct examined students' performance in identifying fractions as numbers (tasks 21, and 32-34) and locating them on number lines (tasks 16-17, and 35-37). Task 19 required students to find a fraction that would be located between two given fractions; in task 42 students were required to identify among a number of

fractions the one that is closer to the number one.

It is also important to note that in Cyprus there is a national curriculum used in all elementary schools. Like curricula in other educational settings (Amato, 2005; Lamon, 1999), the Cypriot curriculum places more emphasis on the part-whole interpretation of fractions. The remaining subconstructs are mainly taught in fifth and sixth grades.

METHODS

The tasks included in the tests were content validated by three experienced elementary teachers and two university tutors of Mathematics Education. Based on their comments, minor revisions were made to the test. The final version of the test was administered to 351 fifth graders and 321 sixth graders (316 boys and 356 girls). To avoid a single test period of undue length, the test tasks were split into two sub-tests, which were administered to students over two consecutive schooldays. Students had eighty minutes to work on each sub-test. To examine whether the tasks used in the tests could form a scale for measuring students' understanding of fractions an Item Response Theory (IRT) model was fit to the data. In the scales developed using IRT models, the task parameters (β : difficulty of the task and α : item discrimination) do not depend on the ability distribution of the examinees and the parameter that characterizes the examinees (θ : ability) does not depend on the set of the test tasks (Hambleton, Swaminathan, & Rogers, 1991). The data were analyzed by using BILOG-MG (Zimowski, et al., 1996). First, the fitting of the data to a single-parameter model was compared to that of a two-parameter model. Whereas in a single-parameter model all tasks contribute equally to the development of the scale, a two-parameter model allows the tasks to differ in their discrimination parameter. Tasks with higher values of discrimination are more useful for developing a measurement scale, since they are better at separating examinees into different ability levels (Embretson & Reise, 2000). Finally, hierarchical cluster analysis was used to cluster the tasks into different groups, according to their level of difficulty.

FINDINGS

The analysis of the data revealed that both a single-parameter and a two-parameter model fit the data well (reliability indices=.88, and .91, for the one- and two-parameter models, respectively). However, since the difference of the $-2\loglikelihood$ index of the two models was statistically significant ($\chi^2=641.35$, $df=43$, $p<.001$), the two-parameter was preferred as a better-fitting model. This meant that the tasks had different discrimination, and therefore, different contribution to the development of the scale. The analysis pursued henceforth was based on a two-parameter model. The psychometric properties of the scale developed by fitting a two-parameter model were

further examined using the test information curve and by examining the extent to which the tasks had different parameters for different groups of students (Differential Item Functioning, DIF) (Embretson & Reise, 2000). Both criteria revealed that the scale had relatively good psychometric properties. In particular, the scale described the construct under examination satisfactorily well (i.e., students' understanding of fractions) with values ranging from $\theta = -3$ to $\theta = +3$. Moreover, only five items had DIF for students of different grade levels or different gender.

The estimates of the two-parameter model that emerged are presented in Table 2 in ascending order, according to the difficulty level of each task. In particular, the estimates for the difficulty and discrimination of each task, alongside their standard error values are illustrated in Table 2. The last column of Table 2 presents the grouping of the 44 tasks of the test into different clusters, according to their level of difficulty. The use of hierarchical cluster analysis suggested that the tasks be grouped into three levels [i.e., the largest decrease in the value of the distance measure of the Agglomeration schedule occurred from moving from a two-cluster (37.63) to a three-cluster solution (17.55)].

Task	Construct	β	$SE_{(\beta)}$	α	$SE_{(\alpha)}$	Level
23	Part-whole	-3.57	0.62	0.36	0.07	1
26	Part-whole	-2.82	0.44	0.38	0.06	1
1	Part-whole	-2.52	0.32	0.52	0.08	1
6	Part-whole	-2.48	0.29	0.63	0.09	1
5	Part-whole	-2.35	0.22	1.03	0.16	1
4	Part-whole	-2.12	0.16	1.33	0.19	1
25	Ratio	-1.73	0.27	0.35	0.05	2
40	Quotient	-1.73	0.30	0.30	0.05	2
28	Part-whole	-1.63	0.16	0.67	0.08	2
13	Ratio	-1.49	0.12	0.8	0.09	2
9	Ratio	-1.21	0.24	0.29	0.05	2
15	Ratio	-1.09	0.18	0.37	0.05	2
30	Ratio	-1.07	0.13	0.56	0.06	2
10	Ratio	-1.04	0.21	0.31	0.05	2
14	Ratio	-1.00	0.09	0.82	0.08	2
2	Part-whole	-0.78	0.09	0.73	0.07	2
11	Operator	-0.60	0.07	0.85	0.07	2
41	Quotient	-0.59	0.15	0.37	0.05	2
22	Part-whole	-0.43	0.09	0.59	0.06	2
16	Measure	-0.34	0.05	1.21	0.09	2
3	Part-whole	-0.27	0.08	0.62	0.06	2
31	Ratio	-0.20	0.11	0.44	0.05	2
43	Quotient	-0.19	0.06	0.94	0.08	2
17	Measure	-0.14	0.05	1.08	0.09	2
24	Part-whole	-0.11	0.17	0.27	0.04	2
12	Operator	-0.05	0.06	0.89	0.07	2
29	Quotient	-0.03	0.19	0.25	0.04	2
27	Part-whole	0.04	0.07	0.75	0.07	2

42	Measure	0.09	0.07	0.71	0.06	2
20	Part-whole	0.13	0.06	0.90	0.07	2
39	Part-whole	0.31	0.07	0.87	0.07	2
38	Ratio	0.80	0.08	0.80	0.07	3
35	Measure	0.81	0.07	0.95	0.08	3
36	Measure	1.06	0.06	1.47	0.13	3
37	Measure	1.07	0.05	1.98	0.19	3
34	Measure	1.11	0.16	0.45	0.05	3
32	Measure	1.16	0.16	0.47	0.05	3
33	Measure	1.24	0.18	0.42	0.05	3
7	Quotient	1.35	0.15	0.56	0.06	3
18	Operator	1.48	0.13	0.70	0.07	3
19	Measure	1.67	0.12	0.93	0.09	3
8	Operator	2.06	0.23	0.51	0.06	3
21	Measure	2.87	0.53	0.25	0.05	3
44	Operator	3.19	0.39	0.72	0.12	3

Table 2: Two-Parameter model estimates for students' understanding of fractions

Three observations mainly arise from Table 1. First, the third column of Table 3 reveals that the task difficulty level ranges from $\beta = -3.57$ to $\beta = +3.19$. Taking into consideration that the students' abilities also ranged from $\theta = -3$ to $\theta = +3$, one could infer that the 44 tasks of the test are relatively well-targeted against the students' ability measures, something that provides further support to the psychological properties of the scale.

Second, columns two, three, and seven suggest that the tasks of the first level (the easiest tasks of the scale) are all related to the part-whole "personality" of fractions. The second level (tasks of medium level of difficulty) consists of a variety of tasks associated mainly with the ratio and the quotient subconstructs of fractions. The tasks examining students' unitizing and reunitizing abilities, as well as those related to common misconceptions on the part-whole notion of fractions were also clustered in the second level. The third level (the most difficult tasks of the scale) is mainly comprised of tasks examining students' construction of the operator and the measure subconstructs of fractions. In particular, the third level consists of all items examining students' understanding of fractions as numbers, as well as tasks related to locating numbers on number lines. It is also notable that all three tasks related to describing a composite function as a fraction or deciding the factor by which a number or a figure should be transformed to get a specific quantity or size of a given shape were rank ordered among the most difficult tasks of the scale.

Third, the tasks used in the test contributed unevenly to the development of the scale. Given that tasks with higher values can better discriminate students into different ability levels, the fifth column of Table 2 shows that all tasks related to number lines (tasks 16-17 and 35-37) were among the most highly discriminating tasks (tasks with values close to or higher than 1). Tasks 4 and 5, which were associated with identifying

a fraction depicted in a continuous model (rectangular and circular, respectively) also had high discriminating values. The tasks on the quotient subconstruct had the lowest discriminating values.

DISCUSSION

The findings of the present study suggest that the criteria proposed in previous studies to examine students' understanding of the multiple facets of fractions can form a basis for developing a scale for measuring students' construction of rational numbers. The scale developed in this study had good psychometric properties, as indicated by several criteria, a result that renders the scale suitable both for teaching and assessing fractions.

The ranking of the 44 tasks of the test according to their level of difficulty indicates that tasks related to recognizing fractions depicted in continuous area or discrete set representations are fundamental for constructing the notion of fractions. Although one might counterargue that this result was due to the fact that Cypriot students are offered ample opportunities to practice this skill, it is important to recall that the *relative* difficulty of tasks included in scales developed using IRT models does not depend on the examinees' abilities and, consequently, on their learning experiences. The fact that tasks 4 and 5 had high discriminating values, as well as that tasks examining students' unitizing and reunitizing abilities and common misconceptions associated with the part-whole interpretation of fractions were ranked in the second level further substantiate the argument that some notions of the part-whole interpretation are more fundamental for developing the concept of fractions. This hypothesis is also supported by a recent study (Charalambous & Pitta-Pantazi, in press) showing that the part-whole subconstruct of fractions is critical for mastering the concept of fractions.

The study also revealed that most of the items related to the measure subconstruct were ranked among the most difficult items of the scale. This finding lends itself to support the claim that students need to make a qualitative leap in their number conceptualization to develop a full understanding of fractions (Lamon, 1999; Stafylidou & Vosniadou, 2004). The tasks related to the operator interpretation of fractions were also ranked among the most difficult items. This result could be attributed to the fact that the operator notion is closely related to the idea of function-transformation which is relatively difficult for elementary school students to comprehend (Behr et al., 1993). Finally, it is important to note that all tasks associated with locating numbers on a number line were among the highly discriminating items, a finding that provides further support to the argument regarding the central role that number lines should have in teaching and assessing fractions (Hannula, 2003, p.17).

Overall, the present study suggests that instruction should not underestimate students' difficulties in constructing the measure and the operator subcontracts of fractions. At the same time, it also indicates that capitalizing on the affordances that several representational models offer, such as the number line, teachers might better scaffold students' construction of the concept of fractions. Future research conducted in other

educational settings could provide further support to this argument. Further work is also needed in cross-validating the scale considered in the present study and developing more tasks on the quotient subconstruct that appears to have the lowest discriminating values.

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THE EFFECTIVENESS AND LIMITATION OF READING AND COLORING STRATEGY IN LEARNING GEOMETRY PROOF

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The reading and coloring (RC) strategy has been verified that it can enhanced incomplete provers' performance to Taiwan junior high students while they had learnt the multi-steps formal geometry proof. In this study, we transfer the RC strategy into whole class regular teaching and explore its effectiveness and limitation. The results show that the learning effectiveness of RC strategy is better significantly than traditional labelling strategy, it enhance the proof quality distribution of multi-steps geometry proof, and it is less-effective to non-hypothetical bridging students .

INTRODUCTION

The learning and teaching of multi-steps geometry proof in Taiwan

The learning content concerning geometric shapes and solids in Taiwan is considerably abundant in the elementary and junior high school. The geometry lessons mainly focuses on finding the invariant properties of kinds of geometric figures and apply these properties to solve or prove problems. The formal deductive approach of argumentation is introduced and become the only acceptable way in the second semester of grade 8, after introducing the congruent conditions of two triangles. Although the manipulative approach is allowed in finding geometry properties, the way of verifying a geometry property is basically deductive. The task of geometry proof in formal lessons can be divided into two phases. In the beginning, the students learn how to apply one property to show a geometry proposition is correct, that is, to infer the wanted conclusion by one acceptable property under the given condition. We name this is a single-step proof. In this phase, if two or more properties are necessary in a proof question, the textbook then divide the whole question into step-by step of single proof task. The second phase start in the final chapter of geometry lessons in the first semester of grade 9. The students learn how to construct a formal deductive proof with 2 or more geometry properties. From chaining the step-by step of single proof into a sequence of proof to proving an open-ended question which 2 or more properties are necessary. That is the so-called multi-steps geometry proof question. It spend about five weeks of regular lessons.

The teaching style in Taiwan junior high school is basically lecturing. Most of the teachers teach geometry lessons by exposition to about 30 students in one classroom. And the geometry proof task is basically treated as writing the reason of a proposition by applying learnt properties.

The Reading and coloring strategy

The reading and coloring (RC) strategy is initially developed for enhancing incomplete provers' performance in geometry proof. The incomplete provers are grade 9 students those who had learnt the chapter of formal multi-steps proof and was able to recognize some crucial elements to prove but missed some deductive process in 2-steps proof test. The RC strategy ask students to read the question, label out the meaningful terms, and then drawing or constructing given conditions and intermediary conclusions on the given figure by coloured pens, where the congruent configurations in same colour. The RC strategy is modified from a typical teaching strategy used in Taiwan junior high geometry lessons which named 'labelling'. The teachers usually use the short segments, arcs or signs to label out the equal sides or angles between subfigures. The RC strategy was developed based on two principles: one is it should provide an operative tool to students for highlight necessary information, and the other one is it should keep teachers' regular teaching style (Cheng, Y. H. and Lin, F. L., 2006).

The function of colour and visual tool for mathematical reasoning is supported by many literatures. Such that Byrne(1847) used coloured diagrams and symbols instead of letters to present the formal geometry proof in 'Elements'. He proposed that the coloured diagrams are easier to understand the formal deductive process of Euclidean proof. The function of this kind of visual aids was mentioned in Mousavi, Low & Sweller(1995) they showed that a suitable visual presentation may integrate all the information necessary in problem-solving task, reduce the cognitive load and increase memory capacity. Stylianou & Silver(2004) find out that the difference between experts and novices when solving advanced mathematical problem is the use of visual representation. The experts always construct an elaborate diagram to include all the literal information and thinking on this diagram.

From our previous studies (Cheng, Y. H. and Lin, F. L.,2005, 2006), the RC strategy is an effective strategy to incomplete provers. Cheng and Lin(2005) showed that the colouring the known information was effective in a highly interactive instruction. This study showed that the intervention of colouring enhance 12/20 of not-acceptable proof in three different unfamiliar 2-steps items to be acceptable. Furthermore, the RC strategy enhance 14/14 of not-acceptable proof items to be acceptable in the non-visual-disturbed multi-steps questions after about 10 minutes of teacher's demonstrating and 23/24 of the items in the delay post test are acceptable (Cheng and Lin, 2006).

The aim of the study

The effectiveness of the RC strategy was verified in our previous studies which focus on incomplete provers. They are all students who had learnt formal multi-steps geometry proof. And these teaching experiments are conducted after the school lesson. We cannot conclude that the RC strategy is an effective learning and teaching strategy to all students in regular teaching. The main purpose of this study is to explore the effectiveness and limitation of the RC strategy in regular Taiwan junior high teaching.

THE PROCESS OF CONSTRUCTING A MULTI-STEPS PROOF

A standard geometry proof question in junior high geometry lessons and tests is of the form ‘Given X, show that Y’ with a figure which the figural meaning of X and Y are embedded in (fig(X,Y)). When a student face to a proof question, the information include X, Y, fig(X,Y), and the status (Duval, 2002) of X (as the premise) and Y (as the conclusion). The proof process is to construct a sequence of argumentation from X to Y with supportive reasons. This process can be seen as a transformation process from initial information to new information with reasoning operators such as induction, deduction, visual judgment... (Tabachneck & Simon, 1996). The acceptable reason in the junior high geometry proof lessons is deduction with acceptable properties. So, we may say that to prove is to **bridge** the given condition to wanted conclusion by acceptable mathematical properties.

Healy & Hoyles(1998) propose that the process of constructing a valid proof involves several central mental processes:(1)students might sort out what is given properties already known or be assumed and what is to be deduced;(2)students might organize the transformation necessary to infer the second set of properties from the first into coherent and complete sequence. Duval(2002) propose a two level cognitive features of constructing proof in a multi-steps question. The first level is to process one step of deduction according to the status of premise, conclusion, and theorems to be used. The second level is to change intermediary conclusion into premise successively for the next step of deduction and to organize these deductive steps into a proof.

In a single step proof question, the process is relatively simple. The student might retrieve a property ‘IF P then Q’ which condition P contain the premise X and result Q contained in Y and finish the proof. We may say this kind of bridging is simple bridging.

The proof process in a multi-steps proof question is much more complex. Since there is no one property can be applied to bridge X and Y. The student has to construct an intermediary condition (IC) firstly for the next reasoning. The IC might be reasoned a step forwardly from X. It is an intermediary conclusion (Duval, 2002) inferred from X as a new premise to bridge Y. Or, it might be reasoned a step backwardly from Y. It is a intermediary premise reasoned from Y as the wanted conclusion to bridge X. So, the first step in a multi-step proof is quite different to the step in a single step proof. The first step in a multi-step proof may be a goalless inferring from X and concluding many reasonable intermediary conclusions. The next step is to go on the bridging process to Y by selecting a new premise from the intermediary conclusions. Or, it may be a backward reasoning from Y and finding many reasonable intermediary premises and the next step is to set up a new conclusion from the intermediary premises and going on the bridging process from X. In fact, this kind of reasoning is lasting before the final step of completing a proof. No matter this kind of reasoning is constructed by forward or backward reasoning, constructing the intermediary condition in a multi-steps proof is essentially a process of conjecturing and selecting/testing. We may say this kind of reasoning process is hypothetical bridging.

In summary, constructing an acceptable geometry proof can be seen as a bridging process from given condition to wanted conclusion with inferring rules. The necessary process includes (1) to understand the given information and the status of these information, (2) to recognize the crucial elements which associate to the necessary properties for deduction, (3) especially in multi-steps proof, to construct intermediary condition for the next step of deduction by hypothetical bridging, and (4) to coordinate the whole process and organize the discourse into an acceptable sequence.

STUDY DESIGN

Transferring the RC strategy into regular teaching

Our previous studies applied RC strategy to enhance incomplete provers' performance in geometry proof. This type of students have learnt the formal multi-steps geometry proof and are able to recognize some crucial elements and construct meaningful intermediary condition. And in our experiments, these students are grouped into individual interview or small group learning. The above conditions are quite different to real classroom teaching.

We design the whole class teaching with RC strategy based on two principles. The first is it should be easily to apply by the teacher. Since it is not easy for them to apply a completely new method in regular teaching, the strategy should be easy to fit into teachers' typical teaching approaches (lecturing). The RC strategy is modified from a typical teaching strategy used in Taiwan junior high geometry lessons which named 'labelling': using the short segments, arcs or signs to label out the equal sides or angles between subfigures. In this study, we ask the teacher use different coloured chalk pen to show the process of proving in the RC strategy on the blackboard: drawing or constructing given conditions and intermediary conclusions on the given figure by coloured chalk pens, where the congruent configurations in same colour.

The second principle is it has to be practicable use to students in all proving task, including taking notes, exercises, homework. In this study, we provide every student a 8-color pen and ask them to use it in all proof task mentioned above.

The samples

A questionnaire with four items are developed and tested as pre-test in 4 classes of grade 9 students before they learn the chapter of formal multi-steps geometry proof. Two of the items are single step and two are multi-steps. The students' performance in these items are coded into acceptable, incomplete, improper, intuitive response, and no response five types according to the coding framework developed in the national survey (Lin, Cheng and linfl team, 2003). The average score of these four classes in the school tests of geometry lessons are considered. Two of these four classes are selected, one for the RC strategy and the other for the traditional labelling strategy, for this study because their performance in the pretest and score of school tests are not different significantly.

The mathematics teachers of these two classes are both experienced teachers. We divide them into two different teaching groups because the teacher of RC class (T1) accepts our suggestion of applying the RC strategy into her regular teaching and the teacher of labelling class (T2) refuses.

The process

In the beginning, we show a demo of the RC strategy by video type and the effectiveness of the RC strategy from our previous study to both teachers. We discuss the function and possible procedure for using this strategy in the five weeks, whole class teaching. Since the teacher T2 refuses to apply the RC strategy in his teaching, we then go on our study with only teacher T1.

During the 5 weeks of teaching the chapter of formal multi-steps geometry proof, T1 uses different coloured chalk pen to show the process of proving: she draws or constructs given conditions and intermediary conclusions on the given figure by coloured chalk pens and uses the same colour in the congruent configurations. At the same time, she asks her students imitate her way in all proving task: taking notes, doing exercises and homework. She checks students' work carefully and ask her students strictly to use this strategy. We provide every student in RC class a 8-color pen. The classroom activities are video typed and students' manuscripts are photographed.

Just after the school test of the chapter of formal multi-steps geometry proof, a post test is conducted for both the RC class and the Labelling class. We compare the leaning effectiveness of the RC strategy in both quality type of proof from the post test and the score from the formal school test. The items of post test are composed with four multi-steps geometry proof questions. These items are used in our previous studies (Cheng, Y. H. and Lin, F. L., 2006). Fig1 is the item used both in the pretest (item 4) and post test (item 1). We use this item to explore what happens from the beginning to the end of the teaching.

A is the center of a circle and AB is a radius. C is a point on the circumcircle where the perpendicular bisector of AB crosses the circle.

Prove that $\triangle ABC$ is equilateral.

Healy and Hoyles(1998)

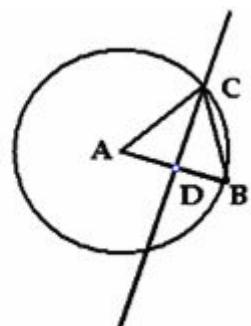


Fig1. The 2-steps item for comparison

RESULTS AND DISCUSSION

The learning effectiveness of RC is better significantly than traditional Labelling

The score of school test after the teaching and the performance in the post test are shown in Table 1. Table 1 show that the score of school test after the multi-steps

lessons and the quality distribution of the post test of RC class is significantly better than Labelling class. The score of school test is significantly better and the percentage of acceptable type in all items is too. This result shows that the RC strategy can be applied in whole class regular teaching. And its effectiveness is significantly better than traditional labelling strategy in both proof quality and formal school test.

class		RC class				Labelling class			
Score of School tests	Average	49.21				43.73			
	Standard deviation	24.54				19.50			
performance in the Post test (percentage)	item	1	2	3	4	1	2	3	4
	No response	0.0	39.4	12.1	6.1	18.2	21.2	9.1	9.1
	Intuitive response	12.1	0.0	0.0	0.0	12.1	3.0	6.1	0.0
	Improper	15.2	15.2	39.4	36.4	9.1	33.3	60.6	39.4
	Incomplete	12.1	6.1	6.1	15.2	30.3	24.2	15.2	24.2
	Acceptable	60.6	39.4	42.4	42.4	30.3	18.2	9.1	27.3

Table 1: The performance of the samples in the pretest

The RC strategy enhance the quality distribution of multi-steps geometry proof

According to Lin, F. L.; Cheng, Y. H. & linfl team(2003), there is about one quarter of Taiwan junior high students, while they finish the formal multi-steps geometry proof lessons, can construct acceptable proof in an unfamiliar 2-steps question. More than one-third of them do not have any response. And approximately one third of them are incomplete. We use this item of the national survey both in pretest (item 4) and post test (item 1). Comparing the distribution of quality type from this study and the result of the national survey (shown in Table 2), we can find out that the percentage of incomplete type in the RC class is significantly less than in the national survey. And there is no more performance of type of no response. This result show that the RC strategy may help many of the ‘potential’ incomplete provers to overcome their learning difficulties when learning multi-steps geometry proof in the traditional teaching.

Furthermore, there is no more performance of type of no response is meaningful. It shows that the RC strategy may help some ‘potential’ no response students to recognize some meaningful information and start to prove. Many students can not start to prove because they can not find out any information associate to a suitable mathematical property. The coloured subfigure may provide more information which is implicit in traditional teaching.

In conclusion, the RC strategy shows the subfigure which associate to a geometry theorem and keeps all information visible and operative. It is helpful to retrieve suitable theorem for reasoning and also helpful to reduce the memory loading when

organizing several steps into a proof sequence. Duval(2002) proposed that retrieving the suitable theorem is one of the key processes in geometry proof and this process is highly depend on the theorem mapping. This study shows that the theorem is easier to retrieve when the correspondent subfigure is highlight by colouring.

	acceptable	Incomplete	Improper	Intuitive response	No response
National sample	24.6	35.0	0.3	2.8	37.4
RC class in the post test	60.6	12.1	15.2	12.1	0.0

Table 2: The distribution of proof quality in the national survey and RC class

RC is less-effective to non-hypothetical bridging students

Even the results in this study show that the RC strategy is more effective than traditional labelling strategy in whole class regular teaching, there are nearly 40% of students can not construct an acceptable proof. In order to investigate the limitation of the RC strategy, we conduct a post analysis. We re-code the manuscripts of the pretest by considering the performance of *hypothetical bridging*. The hypothetical bridging is a necessary process when proving a multi-steps geometry proof. This process construct the intermediary condition (IC) and motivate the second level (Duval, 2002) of proving. We categorize students' performance of the two multi-steps questions in the pretest in to three type: (1)hypothetical bridging, it means that the students construct some intermediary conditions for the next step of reasoning, no matter the IC is useful or not in that question; (2)simple bridging, it means that the students finish the proof by applying only one mathematical property, and it is of course not correct; (3)no reasoning, such that no response, transcribing the item. The performance of these three types of students in the post test is shown in Table 3. Table 3 shows that all the acceptable proof comes from the students who are able to prove a multi-steps question by hypothetical bridging. There are 19/23 of hypothetical bridging students can construct an acceptable proof and no one of non-hypothetical bridging students can do it. It is obviously that the RC strategy can not help the non-hypothetical bridging students to enhance their proof quality more. Since the main function of RC strategy is showing the subfigure which associate to a geometry theorem and keeping all information visible and operative, it is more useful in the information processing process than overcome the cognitive gap. It may help students to retrieve suitable theorem for reasoning and also helpful to reduce the memory loading when organizing several steps into a proof sequence but if the students understanding of geometry proof is only restricted in the first level (Duval, 2002) of proving, that is applying one theorem to bridge the premise and conclusion, then the coloured figure may only help the student to find out the first step (and only one step to him/her) to prove.

	Intuitive response	Improper	Incomplete	Acceptable
No reasoning	1		1	
Simple bridging	3	4	1	
Hypothetical bridging		1	3	19

Table 3: The performance of three reasoning type in the post test

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GRADE 5/6 TEACHERS' PERCEPTIONS OF ALGEBRA IN THE PRIMARY SCHOOL CURRICULUM

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Although there has been a recent push towards having an active “early algebra” curriculum in primary school mathematics classrooms, many curricula still leave formal consideration of algebra until secondary school. Nevertheless, there are many aspects of mathematics in the primary school that prepare students for later algebra study. This research used a questionnaire and interview with 14 Grade 5 and 6 teachers to determine their views and knowledge about algebraic aspects of the primary curriculum. As a group, the teachers had only a limited sense of how the mathematical activities they utilise in the classroom build a foundation for later work in algebra. Furthermore, although they were generally good at recognising the correctness of students’ solutions, they did not seem to engage deeply in the students’ reasoning, and varied in the views of the value to be placed on some responses.

INTRODUCTION AND BACKGROUND

The past decade or so has seen an increased interest in the place of algebraic activities in the primary (elementary) school. This is partly because of the importance to algebra of generalisable structural arithmetical understanding (e.g., the distributive law, the quasi-variables of Fujii & Stephens, 2001), together with the results of studies (e.g., Blanton & Kaput, 2004; Warren, 2005) suggesting that “young children can do more than we expected before” (Lins & Kaput, 2004, p.64).

Bednarz, Kieran and Lee (1996) highlight that three of the basic ingredients of school algebra are the generalisation of patterns (such as number patterns or geometric patterns), the generalisation of numerical laws, and functional situations. Kieran (1996, 2004) groups these together under the heading *generational activities*, because each involves the production of some object of algebra: an equation relating quantities, a description or relation capturing the generality of a pattern, or a rule that describes some general numerical behaviour (see Kieran, 2004, pp.22-24). She points out that while there has been a focus on developing facility with manipulation (something that most adults recognise from their own secondary school algebra experiences, and which many might characterise as *being algebra*), she also emphasises the need for a conceptual understanding of the objects of algebra, and suggests that “noticing structure, justifying and proving have been sorely neglected in school algebra” (p.31). More recently, she has stressed the importance to “early algebra” of analysing relationships, generalising, noticing structure, and predicting, as these are ways of thinking that are foundational for conventional “letter-symbolic algebra” (Kieran, 2006, p. 27).

One class of generational activity sometimes conducted with primary school students involves pattern recognition, often based on a visual design that “grows” iteratively in a sequence. Blanton and Kaput (2004) looked at very young children’s ability to describe functional relationships and found evidence for co-variational reasoning, or keeping track of how one variable changes with respect to another. Warren (2005) found that Grade 4 children are capable of thinking functionally, and can describe—at least in visual terms—how a pattern is generated. Not surprisingly, describing a design according to its position in the sequence is harder than describing the progression from one design to the next. Rossi Becker and Rivera (2006) found similar results with Grade 6 children, and examined how figurative reasoning usually results in greater success than reasoning with numerical quantities alone.

In Australia, where the preparation of primary teachers through university courses usually involves the content and pedagogy of the primary school and little formal study of secondary mathematics, most primary teachers have little expertise in algebra apart from that encountered in their own secondary schooling. With this in mind, and with the belief that primary school students should be engaged in activities that prepare them for algebraic thinking (in all the senses identified by Kieran), we will examine, to a small degree, the algebraic beliefs and knowledge of primary school teachers. The present research looks at two questions: first, what aspects of primary school mathematics do teachers think serve as preparation for high school algebra; and, second, what features do they focus on in students’ responses to a pattern recognition question?

METHOD

This investigation was part of a larger study investigating the pedagogical content knowledge of Grade 5 and 6 teachers in an Australian state. The 14 participants were volunteers, whose teaching experience ranged from 2 to 22 years. The study involved a questionnaire, interviews, and the observation and video-taping of a few lessons. The data considered here came from the questionnaire and follow-up interview conducted at the beginning of the study. The 17-item questionnaire covered a range of mathematical content and pedagogical issues. Teachers responded to the questionnaire in their own time, with no restriction on accessing resources, and the follow-up interview allowed the researchers to probe for elaboration and clarification. The interview questions thus varied from teacher to teacher, depending on their written questionnaire responses and the time available for the interview.

Two questions from the questionnaire are the focus of this investigation. The first asked teachers “What aspects of the primary school mathematics curriculum do you think prepares students for Algebra in secondary school?” The second item, shown in Figure 1, immediately followed the first and provided teachers with an algebraic patterning item, together with sample correct and incorrect student responses. They were asked to indicate how they might respond to the students, and to discuss the role of such items in the primary school.

Data from both the written questionnaire responses and the teachers' interviews have been combined, and are not distinguished except where changes in response occurred. The first question and the last part of the second provide information about the teachers' understanding of algebra's place in the curriculum. Responses to the first part of the patterning question provide insights into the teachers' understanding of algebra itself, and how they recognise and address their students' understanding.

Your students are doing the following exercise on patterns:

Use the matchstick shapes to assist you to complete and extend the grid, then find a rule that will match the grid.



Number of Triangles	1	2	3	4	5	6	7	8	9	10
Number of Matchsticks	3	5								

Here are some examples of rules that students have produced:

Student A: $(\text{number of matches}) = (\text{number of triangles}) \times 2 + 1$

Student B: $(\text{number of matches}) = (\text{number of triangles}) \times 3 + 1 - (\text{number of triangles})$

Student C: $(\text{number of matches}) = (\text{number of triangles}) \times 3 - (\text{number of triangles} - 1)$

Student D: $(\text{number of matches}) = (\text{number of triangles}) + 2$

Student E: Add two more each time

What would be your response to each of these students?

Why do you think these kinds of pattern activities are included in the primary school mathematics curriculum?

Figure 1: Patterning item as presented on the questionnaire.

RESULTS

The primary school curriculum as a preparation for secondary school algebra

The aspects of the primary school curriculum that prepare students for algebra in secondary school that were mentioned by the teachers are shown in Table 1, together with the number of teachers mentioning each one. In some cases it is not clear what teachers meant by certain suggestions, and the interview did not pursue this due to time constraints or a focus on other items. It is conceivable that teachers may have meant "patterning activities" by "problem solving", since that kind of activity often arises under the guise of problem solving, although three teachers specifically mentioned both. "Magic squares" are possibly a particular kind of missing number activity, in which students determine the numbers that belong in the empty spaces of a magic square having constant row and column totals.

Topic area	Number of teachers
Missing number/empty box/cloze problems	7
Patterns (generational activities)	6
Order of operations/properties of operations	6
Problem solving (e.g., with tables/lists)	5
Realising letters can stand for a number	5
Substituting into a formula (e.g., $A = L \times W$)	3
Ratio problems; Understanding of equals sign; Magic squares	2
“Number sense”; Negative numbers; Finding common denominators; Multiplication and division*; Units for area and volume*; Congruent shapes*; Formulating rules	1

Table 1: Number of teachers indicating different primary school topics as preparatory for secondary school algebra. [In responses marked with a * it is not clear what algebraic aspect might have been meant.]

Many of the topics mentioned were what might be expected, although as can be seen no topic was mentioned by more than half of the teachers. The most common response was “missing number” problems, an activity with an “unknown” symbol. Such problems resemble the “solve” symbolic manipulation tasks that many regard as typical of high school algebra, despite the fact that such problems are often solved by arithmetic approaches rather than algebraic ones (see Filloy & Rojano, 1989). Teachers also mentioned pattern generation activities and order of operations as important, with two teachers, who may have experienced similar professional development in pre-algebra ideas, making specific mention of the distributive law.

All but three of the teachers mentioned two or more topics, although only four could list five or more. One of the teachers made no suggestions, writing “I honestly don’t know as I didn’t really do secondary school maths and never had any success at maths in secondary school”. Another of the teachers with a limited response indicated that she felt the curriculum introduces algebra in secondary school and so very little is done at primary school level. In her interview, however, she expanded on the importance of functions and number patterns as a transition to secondary schooling. She also commented about a question we had included on a student quiz, which involved generalising a pattern in order to predict the 100th term. She said “Probably I wouldn’t go to what’s a hundred of that. I might go to five or something, so they can work it out, but not necessarily that generating patterns, which I think is something I’d like to explore.” This suggests she may have a limited awareness that asking for big-numbered terms can force students to generalise properly, rather than just iterating through the first few terms.

Responses to the second part of the patterning question provided further insight into teachers' perceptions of the purpose of this kind of activity. Nine of the 14 teachers explicitly stated that such activities lead to algebra, with one of these suggesting that it can help with the introduction of symbols. Two of these nine also emphasised affective issues, indicating that if students encounter such activities in primary school they will not be as "fearful" about algebra in high school. Nine of the teachers (six of whom had mentioned algebra) viewed pattern recognition and description as important for mathematics or real life in general, listing one or more of generating, creating, seeing, or developing rules for patterns as significant skills for students. Six of the teachers also emphasised the thinking skills that such activities develop, with another three mentioning that the activity can develop problem-solving skills.

Although the teachers were not asked about the extent to which they conducted patterning activities in their own classrooms, most of the teachers' comments implied or explicitly stated that they used such tasks. One of the Grade 5 teachers, however, said that although she had seen this kind of question she had not used them. Some teachers also commented on their students' engagement and facility with these activities. One wrote "I don't think it is a particularly easy concept to develop and children need time to 'play' with ideas and understandings". Three of the teachers made specific reference to higher ability children, with one saying

Because I think that that's what makes the brainy kids. That's what makes the kids who are very good at maths, I believe that they're actually fantastically fast at picking up the pattern, using patterns they've used previously, and applying them.

Algebraic understanding evident in responses to the patterning item

We now examine the understanding of algebra and student reasoning evident in teachers' responses to the first part of the patterning item. From the questionnaire results, eight of the teachers correctly recognised the erroneous response from Student D, with a further three realising the error in the interview, in one case prompted by the interviewer. Two of the teachers did not comment on correctness, and the final teacher—who had not listed any pre-algebraic aspects of primary school mathematics in response to the first question—indicated that he thought "all of these students are correct in their own way". He said that he liked D, but added that "with all of them they need to explain them to me".

The teacher who had commented about students needing time to develop facility with pattern activities was one of the very few who gave a detailed description of how she would respond to the students, which she did in such a way that she clearly revealed how well she understood their reasoning. Her responses were written as if she was talking to the student concerned:

A: Yes, this formula appears to work. Well done! How did you come up with it?

B: Yes, this formula does work. Is there an easier way of writing it? If you are multiplying by 3 and then later subtracting the number of triangles, isn't it the same as saying (number of triangles) $\times 2 + 1$?

C: Yes, this formula does work. There are quite a few steps to it. Is there an easier way of writing it?

D: While this formula works for the first triangle it does not work for the rest of the pattern. Let's test it out. I think you have created this formula because adding a new triangle to the pattern requires two more matchsticks. So, in theory it sounds like it should work. How could we use multiplication in our formula?

E: Yes, that will work if you know what the preceding number is. However, what if we want to work out how many matchsticks are used to make 23 triangles? I think we need to find a quicker formula that does not mean we have to work out the preceding 22 triangles.

Only one other teacher really indicated an understanding of the thinking that might have led to Student D's response. Nobody else showed serious engagement with the details or possible derivations of the formulae produced by the students—with the exception of E—despite most stating that A, B, C and E work. Presumably they checked the outputs of the formulae against their own data. All of the teachers gave some indication of which responses they thought were “better” than others, although they varied in what they meant by “better”. Nine of them commented about responses B and C being “complicated” or “less efficient” in comparison to A, and expressed a hope that students—or at least their more able students—would be able to find an “easier way”. One of them, however, actually rated B and C more highly *because* of the complexity. Only one teacher said that she valued having multiple solutions and that “you need to be able to see that they're all valid, logical paths”. She did not, however, suggest that exploring the *equivalence* of such solutions might be significant (see also APPA Group, 2004).

Six of the teachers commented that they liked the fact that Student D, though incorrect, had tried to express a rule in a relational way, with some of them even preferring this to Student E's correct, but non-equational, response. In fact, reactions to Student E's solution were polarised, with some regarding it as good and others as poor. The eight teachers who seemed to regard it negatively commented that it was “too simplistic”, that the student “can identify a pattern, but they're not necessarily patterns within relationships [...], they need to relate it to the number of triangles”, or that the student “hasn't taken time to find a rule that is more detailed”. One of the teachers suggested that Student E would struggle with more difficult problems, although he has “got the basics”. In contrast, other teachers felt it was good that E had spotted the pattern so quickly, and that it was “nice and simple”. In fact, at least one of the teachers actually seemed to struggle herself with the covariational ideas expressed in responses A, B and C, and appeared much more comfortable with Student E's solution. Only four of the teachers actually explained the limitations of Student E's response in terms of the relationship being given iteratively. They pointed out that you cannot work out the answer for a given number of triangles without knowing the preceding term.

Finally, in light of the work of Rossi Becker and Rivera (2006), we note that none of the teachers' comments suggested an appreciation of the fact that a formula or description of a relationship usually depends on the way the pattern is perceived.

DISCUSSION AND CONCLUSION

Before discussing the results, we must note there were many other questions on the questionnaire and interview, and that these items were late in the questionnaire. This may have affected the quality of the teachers' responses. Furthermore, the items and questions did not specifically target some of the aspects we have discussed, such as the effect of perception on pattern description. The results should be viewed with these limitations in mind, although we can still draw some important conclusions.

Among these teachers, at least, there is some awareness of the kinds of algebraic ideas that can be fostered in the primary school. Unfortunately although all but one could contribute at least one reasonably strong pre-algebraic concept (missing number problems, patterns work, order of operations, importance of the equals sign, realising letters can stand for a number, or formula substitution), only six could list three, and the fact that no more than half of them listed any one of the topics is also of concern. The only time teachers explained *how* a given aspect contributed to algebra learning was in the fairly obvious case of the missing number problems.

For some teachers, this limited perspective may be due to their own educational history, as evident in the explanation from the teacher who could not list any content areas. One of the older teachers, who explained that her mathematics teaching had undergone a transformation as part of professional development she had undertaken, explained that as far as the patterning item was concerned that "I've not had a lot of experience with that sort of thing, but I think it's something that we really need to get the kids doing ... I didn't get that sort of thing when I was at school".

There are positives and negatives to note in the teachers' engagement with the patterning item, too. Most were able to recognise the correctness or otherwise of the students' responses, and could offer reasons for their judgements of which responses were better. What was missing, though, was a deep engagement in the mathematical and algebraic underpinnings of the activity. Again, this may be a consequence of educational background, and the experience and training the teachers may have had (or, more likely, may *not* have had) in conducting this kind of task.

Given the power of pattern recognition and similar generational activities for developing facility with algebraic concepts—whether noticing structure, generalising, developing functional reasoning, using symbols, or even manipulating symbols—it seems that more needs to be done to help teachers understand what the key aspects are and how they contribute to the understanding that needs to be developed in the secondary school. It is also evident that teachers may need more guidance to help them use generational activities in an effective way in the classroom. This would allow them to make better judgements about the correctness, derivation, value, and problematic

aspects of different approaches that students might take, and provide better assistance to students as they engage with the task.

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THE INFLUENCE OF INQUIRY-BASED MATHEMATICS TEACHING ON 11th GRADE HIGH ACHIEVERS: FOCUSING ON METACOGNITION

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This study investigates the influence of inquiry-based mathematics teaching on high achievers' metacognitive abilities. The research subjects were 28 eleventh graders high performance in learning mathematics. A mixed methodology combining qualitative and quantitative approaches was used to investigate students' metacognition in an inquiry-based classroom environment. The main research instrument for collecting quantitative data was the Metacognition Inventory questionnaire which was conducted before and after the inquiry-based lessons. The qualitative data, such as interviews with students, videotaped classroom teaching, students' work sheets and feedback sheets, and teacher's journals, were also collected and analysed. Results indicated that there was weak but no significant correlation between student's mathematics achievement and their metacognition. Besides, students could develop significantly better metacognitive capacities after receiving the three-month inquiry-based mathematics teaching. Moreover, we also discussed how inquiry cycle is related to metacognitive components.

INTRODUCTION

How teachers can help students participate in the process of knowledge construction has been a central issue in the debate on mathematics education (e.g., Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Inagaki, Hatano & Morita, 1998; Lampert, 1990). Within the contribution to this debate, educators and researchers are convinced that students need to have ample opportunities to progress from concrete to abstract ideas, rethink their hypotheses and adapt and retry their investigations and problem solving efforts (Hinrichsen & Jarrett, 1999). Teachers ought to provide certain kinds of experiences in which students are able to work as young mathematicians or researchers. In short, students should develop their mathematical knowledge through inquiry-based teaching. The inquiry teaching methodology is built on the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) and the report of *Project 2061*'s benchmarks (American Association for the Advancement of Science [AAAS], 1993). They both assert that inquiry as a high-quality teaching to engage students in the processes of learning and creating mathematics and also recommend that students should have the ample opportunities to utilise inquiry cycle in carrying out their own mathematical investigations when learning mathematics. The essential traits of inquiry that can be generated from some reports (e.g. AAAS, 1993; Hinrichsen & Jarrett, 1999) are concluded as follows:

connecting former knowledge and experiences with the problem as learners have, designing procedures (plans) to find an answer to the problem, investigating phenomena through conjecture, constructing meaning through use of logic and evidence and reflection.

Metacognition takes on importance in mathematical classroom because research evidence has shown that enhancing students' metacognition could lead to corresponding improvements in learning outcomes (e.g., Baird & Northfield, 1992). The concept of "metacognition" is defined as one's knowledge concerning one's own cognitive processes and awareness of a mathematical problem that involves the process of planning, monitoring and evaluation of a specific problem solution (Flavell, 1976, 1992). Following Flavell's studies, researchers investigated many aspects of metacognition in mathematics education, such as Schoenfeld's work (1987, 1992) on comprehensive analysis of metacognitive processes in problem solving, which put more emphasis on mathematical thinking and problem solving processes. Recent studies appeared some major common elements which can characterise good instructions that enhance students' metacognition (e.g. Kramarski, Mevarech & Arami, 2002; Mevarech & Fridkin, 2006). These instructions should focus on: (a) comprehending the problem; (b) constructing connections between previous and new knowledge; (c) considering strategies appropriate for solving the problem; (d) reflecting on the processes and solution. By means of analysing the essential traits of inquiry compared with these suggested instruction focuses, it seems reasonable to hypothesise that inquiry-based teaching may promote students' metacognition.

As we mentioned above, research has proven high metacognition could produce high achievement. However, Alexander, Carr and Schwaneflugel's (1995) indicate that research does not support the viewpoint that high achievers have vastly better or more advanced metacognitive abilities in all areas of metacognition, but it appears that high and low achievement children are equally capable of using some metacognition. The result of International Assessment of Educational Progress [IAEP], a large-scale international achievement survey, reveals that Taiwanese students were with high performance in mathematics but weakness in higher order thinking skills. The main reason might be most schools in Taiwan tend to remain a conservative pedagogy with behaviourist paradigms to teach mathematics. Therefore, this study is designed to explore how the inquiry-based mathematics teaching influences on high achievers' metacognition developing.

THEORETICAL FRAMEWORK

Inquiry-based mathematics teaching

The inquiry based-teaching could be supported by the use of inquiry cycle (Siegel, Borasi & Fonzi, 1998). Lawson, Abraham, and Renner (1989) proposed an E-I-E (Exploration, Invention, and Expansion) inquiry cycle which has long term been considered within inquiry teaching and modified or refined into various frameworks. In Exploration phase, it should provide students with the opportunity to bring out prior

knowledge, explore a range of phenomena for themselves, and experience a confrontation to their own way of thinking. In Invention phase, it should help students organise their information from the Exploration phase. Besides, the teacher should consider how the idea or skill is modelled or demonstrated. In Expansion phase, the goal is to help students finish restructuring old beliefs, old knowledge structures and it is also important to help students apply and transfer the new idea to new situations. Although there are some later frameworks of inquiry cycle using different terms to elaborate their structures, the basic theoretical backings still seem to fit under the core ideas of the E-I-E model. Therefore, for simplifying and effectively applying the inquiry cycle, we do not consider the later frameworks but adopt E-I-E model as the main approach to address our inquiry-based teaching.

Metacognition

Flavell (1979) defines metacognition as “thinking about thinking”, and elaborates metacognition as (i) awareness of how one learns; (ii) awareness of when one does and does not understand; (iii) knowledge of how to use available information to achieve a goal; ability to judge the cognitive demands of a particular task; (iv) knowledge of what strategies to use for what purposes. He also distinguishes between two components of metacognition: (a) knowledge of cognitive processes and products; and (b) ability to control, monitor, and evaluate cognitive processes. Flavell argues that knowledge of cognition depends on the following inter-related components: metacognitive knowledge about self, the task and strategies; knowledge about how to use the strategies; and metacognitive experience. The later refers to one’s feeling about being successful (or unsuccessful) in performing the task. According to this model, the metacognitive knowledge leads to strategy use which in turns affects the metacognitive experience that affects the acquisition of metacognitive knowledge and so on. Corresponding to Flavell’s focusing on metacognitive knowledge, Brown (1987) outlines metacognition as (i) an awareness of one’s own cognitive activity; (ii) the methods employed to regulate one’s own cognitive process and (iii) a command of how one directs, plans and monitors cognitive activity. Stating differently, metacognition is made up of active checking, planning, monitoring, testing, revising, evaluating, and thinking about one’s cognitive performance.

METHODOLOGY

Subjects

The research subjects of this study were twenty-eight 11th graders (all girls) who were selected from the whole grade with the best performance in mathematics in a famous girl’s senior high school in Taiwan. These students were graded within 93% in mathematics in the national high school entrance examination.

Data collection

For the quantitative data, we collected students’ average mathematics scores in last academic year (10th grade) and student’s performance on Metacognition Inventory

questionnaire [MI] (adopted from Chang, 1994) which was administered at the beginning (pre-test) and end (post-test) of the three month inquiry-based teaching practice. The instrument, MI was designed to assess student's metacongition with six subscales: (i) Selective Attention (SA): deciding to attend to specific aspects of input; (ii) Organising (O): the act of rearranging the information which one gets from SA; (iii) Strategising (S): planing or using appropriate strategies in solving the problem; (iv) Self-Testing (ST): assessing how much one understands by self-questioning; (v) Self-Monitoring (SM): the activities that moderate the current progress of learning; (vi) Self-Correction (SC): correcting errors and implementing remedial or changing strategies. The MI comprised 48 items distributed across the six subscales (8 items per subscale) on a 4-point Likert scale reflecting student's metacognitive behaviours as "1" means strongly disagree, whilst "4" means strongly agree. Furthermore, this questionnaire was piloted with 138 alternative high achievers from the same school and obtained acceptable Cronbach's α coefficients as follows: SA, 0.665; O, 0.797; S, 0.736; ST, 0.723; SM, 0.732; SC, 0.771; total scale, 0.927.

On the other hand, for the qualitative data, we collected videotaped classroom teaching, teacher's journal, students' semi-structured interview results, and feedback sheets after each class. These qualitative data were analysed to explain or elaborate the quantitative results for interpreting the relationship between inquiry-based teaching and student's metacognition growth. In particular, the semi-structured interviews were conducted individually to the subjects who were considered as worth of further investigation for 15-30 minutes after each inquiry-based lesson (totally five) and also after they completed MI in the beginning and the end of this study.

Procedures

Strictly speaking, this study spent almost two years which was from January 2005 to December 2006. The whole research can be categorised into three periods: (a) the first period (January 2005 ~ February 2006) serves as a warm up and preparation for the participant teacher implementing inquiry-based teaching. The teacher joined a university-based professional development project which was funded by National Science Council in Taiwan. This project aimed to enable in-service teachers to apply inquiry-based mathematics teaching in their classrooms. After over one year of participation, the participant teacher was recognised having enough skills to teach through inquiry. (b) In the second period (February ~ August 2006), the participant teacher collaborated with a mathematics educator and a group of mathematics teachers to design curricula that are suitable for inquiry-based teaching. Meanwhile the participant teacher also started to introduce collaborative learning in his classroom for encouraging students to engage in sense-making and discussions. Because literature indicates that inquiry method is high correlated with collaborative learning strategies, taking place collaborative leaning could enhance inquiry-based teaching. (c) In the final period (September ~ December 2006), inquiry-based teaching was conducted in the subject class. The students answered the MI as pre-test and post-test before and

after this period. All the qualitative and quantitative data were also collected during this period.

RESULTS AND DISCUSSION

Table 1 reports the relationship between students' mathematics average scores in last academic year and the scores in the pre-test of the MI. Analysis of the data showed that there was weak but no significant correlation ($r=.201$, $p=0.15$) between student's mathematics achievement and their metacognition. The only significance appeared in Selective Attention ($r=.334$, $p< 0.05$) and it remained weak. These results may imply that the linear relationship between mathematics achievement and metacognition did not occur with these high achievers. It seems reasonable to claim that these high achievement students did not correspond to high metacognition capabilities. This claim might be consistent with the typical Taiwanese high achievers' performance as mentioned earlier in this paper. In addition, based upon their initial MI scores ($M=2.66$ in pre-test, see Table 2) and the qualitative data we collected, our claim seems to be confirmed:

T: Although my students could obtain high mathematics test scores, they did not perform well in dealing with open tasks; most students could only apply straightforward and limited strategies. They weakly showed metacognitive awareness. (20060910 teacher's journal)

S12: I thought I am lack of this kind of ability (metacognition). I am not good at organising the problem information and pursuing my thinking approaches. I think it is better that the teacher can directly talk about what I don't know. (20060901 interview)

	SA	O	S	ST	SM	SC	Total
MA (N=28)	.334*	.162	.069	.216	.021	.209	.201

1. MA: Mathematics Achievement, SA: Selective Attention, O: Organising, S: Strategising, ST: Self-Testing, SM: Self-Monitoring, SC: Self-Correction
2. *, $p< 0.05$

Table 1: Correlations between mathematics achievement and metacognition subscales

Table 2 exhibits means and standard deviations on MI (including sub-scales) of both pre-test and post-test. The difference of total scale between the pre-test and the post-test was highly statistically significant ($t=4.56$, $p< 0.001$). In other words, students performed better metacognition including all components in the post-test than in the pre-test. Subsequent analysis of subscale scores also revealed highly significant changing from pre-test to post-test. This result appears to suggest that inquiry-based teaching was effective in helping students' development of metacognitive abilities.

S23: I feel I'm better on understanding the problem because the new teaching method is not only interesting to me but also helpful to think more deeply. (20060915 feedback sheet)

S10: I was used to waiting for teacher's solution but now I find more chances to explore and analyse the problem by myself and our team. (20061109 interview)

S1: In inquiry teaching, I have more opportunities to try different approaches to a problem and discussion which can also inspire me with new ideas. (20061109 interview)

(N=28)	Pre-test	Post-test	T	p
	Mean (SD)	Mean (SD)		
A	2.91 (.35)	3.12 (.28)	3.35	.002**
O	2.64 (.41)	2.89 (.39)	3.04	.005**
S	2.89 (.42)	3.17 (.41)	3.17	.004**
ST	2.25 (.50)	2.61 (.40)	4.60	.000***
SM	2.56 (.43)	2.97 (.45)	4.43	.000***
SC	2.73 (.52)	2.94 (.44)	2.56	.016*
Total	2.66 (.36)	2.95 (.32)	4.56	.000***

*, p< 0.05; **, p< 0.01; ***, p< 0.001

Table 2: Means and standard deviations of pre-test and post-test

Comparing all subscale's scores in the pre-test, Self-Testing appears the lowest scores ($M=2.25$) within these six sub-scales. Although it significantly increased in post-test ($t=4.60$, $p< .001$), it still remains as the lowest over all subscales in the post-test ($M=2.61$). It seems that students were less likely to examine how much they already know or to determine whether they truly understand. This finding was also parallel to the research which indicates that this kind of regulated ability develops slowly and is quite poor in children and even adults (Pressley & Ghatala, 1990). In further analysis of the items within this subscale (Table 3), we found that the items 2 and 5 show lower mean scores in pre-test ($M_{item2}^{pre}=1.25$; $M_{item5}^{pre}=1.50$) and post-test ($M_{item2}^{post}=1.82$; $M_{item5}^{post}=1.96$). This result might indicate students were less likely to assess what they know by questioning themselves. The following quotation of the teacher's journal could be a note:

T: Students are used to solving given problems but have fewer experiences to pose problems by themselves. I think this is the reason why they obtain lower ST scores. However, in inquiry-based teaching, they could have more opportunities to generate new problems. This may encourage them to ask problems for checking what they have learned. (20061211 teacher's journal)

Moreover, the post-test score of item 4 ($M=2.00$) is slightly lower than the pre-test ($M=1.96$). Unexpectedly, it seems that students were less likely to examine their understanding by extra mathematics problems, both before and after the inquiry-based teaching. The reason might be that they would rather focus on fewer problems which seem worth undertaking than practice too many routine problems. The following response of a student could offer some evidence:

S6: It's great when we work together to investigate a worth undertaking problem. I can take advantage from discussing with my team members. I suggest not to work on too many problems but to have more time on thinking or discussing with others. (20061031 feedback sheet)

Items within Self-Testing (Total item=8; N=28)	Pre-test Mean (SD)	Post-test Mean (SD)	t	p
2. I quiz myself about maths problems in order to check how much I understand	1.25 (.44)	1.82 (.72)	4.08	.000***
4. I solve additional maths problems after completing the exercises in the textbook	2.00 (.94)	1.96 (.64)	-0.21	.832
5. I practise problems to simulate the school test before it comes	1.50 (.64)	1.96 (.58)	3.10	.004**

*, p< 0.05; **, p< 0.01; ***, p< 0.001

Table 3. Means and standard deviations on Self-Testing of pre-test and post-test

In addition, after analysing all the qualitative data, we also tried to identify how the inquiry-based teaching was related to metacognition subscales which occurred in this study. Some results might be concluded as follows: “Selective Attention”, and “Organising” seemed to take place during the Exploration phase of the inquiry cycle; “Strategising” and “Self-Testing” appeared in the Invention and Expansion phases; and “Self-Monitoring” and “Self-Correction” happened across all the phases. For example, when students engaged in the lesson “Circle and Sphere”, they needed to recall their prior knowledge about circle (Exploration-O), focusing on how circle can be defined and their properties (Exploration-SA), discussing and modeling their thinking (Invention-S & ST), applying their conclusion to Sphere (Expansion-O & S).

CONCLUSION

Enhancing students’ metacognition is not a straightforward process by any means. A key factor influencing students’ propensity to enact their metacognitive capacities is their perception of the classroom environment, including how they are taught and also the broader culture within the classroom. In this study, we constructed an inquiry-based learning environment for the students who were asked to personally construct their own understanding by posing questions and considering how investigations will proceed and how findings are analysed and communicated (Hinrichsen & Jarrett, 1999). The results support that these arrangements are meaningful in stimulating student’s metacognition growth. Although there seems no enough direct evidence to prove that the inquiry-based mathematics teaching is a warranty for student metacognition development, it still can be a useful guide for helping teachers to make the most of metacognitive learning experiences for students. Therefore, we might be able to argue that the inquiry-based mathematics teaching method may serve as a catalytic metacognitive experience that informed students about what was for some an alternative conception of learning.

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THE EFFECTS OF “SPATIAL GEOMETRY CURRICULUM WITH 3D DGS” IN LOWER SECONDARY SCHOOL MATHEMATICS

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This research explores the effects of spatial geometry curriculum utilizing 3D dynamic geometry software (3D DGS) in lower secondary school mathematics. The analysis was done by comparing the results of experimental groups with that of control groups in a school (Japan), as well as results of the national survey of Japan. In conclusion, positive effects were identified regarding the construction of spatial figures by moving a plane figure and the explanation for a spatial figure represented on a plane, along with multiplier effects recognized in the relationship between cognitive and affective aspects.

INTRODUCTION

Dynamically manipulable interactive graphical representation of technology contributes to the teaching and learning of geometry. Drag mode is a key element of Dynamic Geometry Environments (DGEs). For example, researchers stressed the key role of dragging in forming mathematical conjectures and categorized different kinds of dragging (Healy, 2000 etc.).

Until now, 2-Dimensional DGEs have been the target of research, and researchers have suggested that the dynamic function of computer software makes its use especially powerful (Cuoco & Goldenberg, 1997; de Villiers, 1998; Goldenberg & Cuoco, 1998, etc.) However, researchers have claimed that students looked at diagrams differently from the way their teachers intended them to (Yerushalmy and Chazan, 1990, etc.). We need to take into account that students often “walk around” their own world in a mathematical situation when we design student-centered learning environments.

On the other hand, Laborde et al. (2006) pointed out the following with regard to overall technology utilization, “In most countries technology is not yet fully adopted by teachers. As a consequence there is very little research that has been done on geometry curricula that start from scratch with technology” (p. 290). To improve this situation, environments that the teacher can more easily adapt have also been provided. For example, in Japan, Iijima has developed a website¹⁾ where interesting problems and situations using 2D DGS “Geometric Constructor” for students are presented. Some of that content was developed through discussions with users.

In addition, regarding DGE generally, it remains unsettled questions such as “The design of adequate tasks” and “How does teacher manage the use of technology in

taking into account the curriculum" (Laborde et al., 2006, p. 296). Particularly with 3-Dimensional DGEs, although 3D DGS have been developed, curriculum development considering "epistemological impact" (Balacheff & Kaput, 1996, p. 469) has not been conducted, nor the effect of the developed curriculum been verified.

The purpose of this paper is to explore the relationship between cognitive and affective impact of spatial figures in a "SPATIAL GEOMETRY" curriculum that utilizes 3D DGS.

METHODS

"SPATIAL GEOMETRY" Curriculum

"SPATIAL GEOMETRY" curriculum²⁾ is composed of "Construction of 3D figures" (six lessons in total), "Cutting of 3D figures" (five lessons in total), and "Surface Area and Volume of 3D figures" (three lessons in total). It is supported through the context of "Let's make stamps cutting available materials". First, students look at various types of stamps and get motivated by thinking "I want to make one all by myself". Then he/she discovers and verifies the characteristics of spatial figures which the stamps are composed of. In addition, the students do all kinds of things to figure out what kinds of spatial figures are adequate to make various kinds of stamps by cutting. In this way, the curriculum has the intended contents and activities embedded within it and takes into consideration the complementation of vertical and horizontal mathematization.

"SPATIAL GEOMETRY" curriculum is characterized as follows from the (a) learning environment, (b) learning content, and (c) learning activity points: (a) Students can use the real things (such as "Polydron" and solid models, etc.), sketches, and 3D DGS interactively while comparing the results of them. In addition, one computer per pair of students is prepared for them. (b) Connections between the mathematical contents and daily life or other areas of school mathematics are enhanced. (c) Activities with "the concept formation of spatial figures", "logical thinking", and "representations of spatial figures" are embedded respectively. For these activities, the use of the 3D DGS will facilitate the three following possibilities.

- The possibility of exploring spatial figures those are physically difficult to construct or operate.
- The possibility of multilaterally observing the process of construction or operation of 3D figures in the dragging.
- The possibility of cultivating logical thinking about spatial geometry with dynamic transformations and multilateral observations.

Questionnaire

The questionnaires are composed of questions regarding students' recognition and consciousness of spatial geometry and its learning.

To collect quantitative data on students' competencies (cognitive aspects) regarding spatial geometry, we used questionnaires which were designed for comprehensive surveys on the implementation of the Japanese national curriculum. These surveys are known as "Kyoikukatei-zishijykyo-chosa" (2001 & 2003) in Japanese. From here on in, these surveys will be referred to as "National survey 2001 (or 2003)". In the national survey 2001 or 2003, target classes were randomly selected with stratified sampling from all over Japan. In addition, each of the surveys was conducted in February of those years. More specifically, among the questions regarding students' recognition, this paper covers two questions. One is the question describing that a spatial figure is composed of the movement of a figure in the plane (2003: No. 1A13 (1)) and the other is the question that captures the length of a line segment in a sketch of cube and states the reasons for that (2001: No. 1B12, and 2003: No. 1B12).

On the other hand, to collect quantitative data on students' emotional experiences (emotional aspects) regarding the learning of spatial geometry, we designed a specific questionnaire. And more specifically, of the questions regarding consciousness, this paper covers the part that students reflects the construction and the cutting of spatial figures respectively from three points of view ("understanding", "interest level", and "usability").

Participants and Procedure

Targeted classes belonged to the seventh grade at a public junior high school in Japan, and at the point of before conducting the curriculum there are no significant difference statistically regarding scores on regular examinations between classes that followed a "SPATIAL GEOMETRY" curriculum using 3D DGS (Experimental group: n=66) and a class that was compliant with the Japanese national curriculum announced in 1998 (1998CS) (Controlled group: n=32). The "With 3D DGS" and "1998CS compliant" classes were compared using the questionnaire method. In addition, survey results from the 3D DGS classes and the national survey are compared.

Lessons were implemented from Jan 23 to Mar 6, 2006 (16 lessons in total), and a survey via questionnaire was conducted on Feb 28, 2006 after the Lesson 12 in "SPATIAL GEOMETRY" was implemented (on Feb 24). Moreover, in the 3D DGS classes, various qualitative data were collected based on research field notes, VTRs, and worksheets.

RESULTS AND DISCUSSION

Construction of a Spatial Figure by Moving a Plane

Survey Question 1 (2003: No. 1A13 (1)) "You want to make a cylinder by moving a plane figure. What kind of plane figure do you use and how can you move it? Explain how in the column □. If necessary, you can also use diagrams."

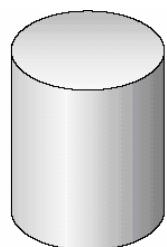


Table 1: Construction of a cylinder with different groups

Group	Pass	Fail	Total
Classes in a same School			
With 3D DGS (n=66)	86.4%	13.6%	100
1998CS Compliant (n=32)	56.3%	43.8%	100
National survey 2003 (n=3617)	56.2%	43.8%	100

Table 1 shows results of the **Survey Question 1**. The pass percentage of “With 3D DGS” (86.4%) was 30.1 points higher than that of “With 3D DGS” (56.3%). The Pass/Fail number of different groups regarding this question was significant, $\chi^2 (1, N=98) = 10.881, p < .01$. Similarly, pass percentage of “With 3D DGS” (56.2%) was 30.2 point higher than that of the national survey 2003 (56.2%), and the Pass/Fail number was significant, $\chi^2 (1, N=3683) = 24.057, p < .01$. Therefore, it can be seen that the curriculum with 3D DGS heightens the pass percentage compared to that was compliant with the 1998CS.

If attention is focused on response status to the answers, the percentages of students who show a development or explain using a development of cylinder (type 4) was 0% for “With 3D DGS”, which is notably low when compared to “With 3D DGS” (28.1%) or the national survey 2003 (11.7%).

In the curriculum with 3D DGS, it was also significant that multiplier effects can be recognized in the relationship between cognitive and affective aspects. For example, the percentage of students who passed this question and positively answered “Did you understand spatial figures constructed by moving line segments or planes?” was 84.8% (n=66) in the “With 3D DGS”. On the other hand, in the “1998CS compliant” it was 53.1% (n=32). Similarly, the percentage of the students passed this question and positively answered “Did you like studying spatial figures constructed by moving line segments or planes?” was 86.4% (n=66) in the “With 3D DGS”. On the other hand, in the “1998CS compliant” it was 40.6% (n=32). In addition, the percentage of students who passed and positively answered “Do you think the fact that spatial figures constructed by moving line segments or plane figures will be useful in daily life or in the business world?” was 69.7% (n=66) in the “With 3D DGS”. On the other hand, in the “1998CS compliant” it was 37.5% (n=32).

It seems reasonable to conclude that these effects were generated by activities embedded within the curriculum. In the curriculum, for example, the students moved plane figures in parallel using 3D DGS to construct prisms and cylinders, and then observed the process and result of those construction from various angles (Fifth lesson: on Jan 31). Next, the students referred to the real things along with dynamic transformations in 3D DGS and drew a sketch of various prisms and cylinders corresponding with the construction of them by moving a plane figure in parallel (Sixth lesson: on Feb 1). The students also used 3D DGS to construct a torus or solids of revolution which are constructed when a plane figure rotates around a axis which does not exist on the plane, and then observed those operations from various perspectives

(Seventh lesson: on Feb 2). Moreover, using 3D DGS, the students compared the construction of a cone by rotating a right triangle and the side of cone by moving a line segment (eighth lesson: on Feb 3). It seems that the students began to understand the construction of spatial figures through these activities. In addition, as the activities relate to the representation of spatial figures, students explained similarities between prisms and cylinders by moving a plane in parallel based on positional relationships of the sides or faces (Fifth lesson), described the necessary ingenuity or precautions taken when they drew a sketch of prisms and cylinders (Sixth lesson), and explained similarities between solids of revolution (Seventh lesson). It seems that these activities helped to increase the abilities of students to verbalize constructions of spatial figures.

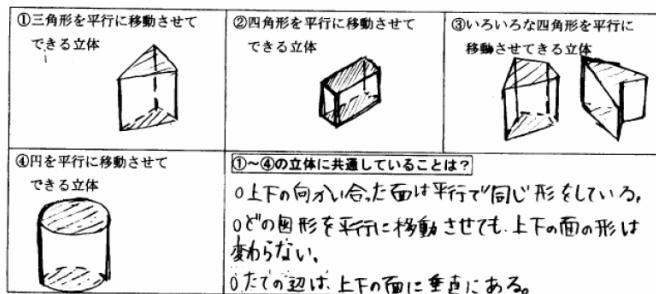


Figure 1: A student's description on similarities between prisms and cylinders by moving a plane in parallel (Fifth lesson)

“The planes on the top and bottom are parallel and a uniform shape. The shapes of the top and bottom do not change when any figure is moved in parallel. The longitudinal sides are vertical to the top and bottom plane.”

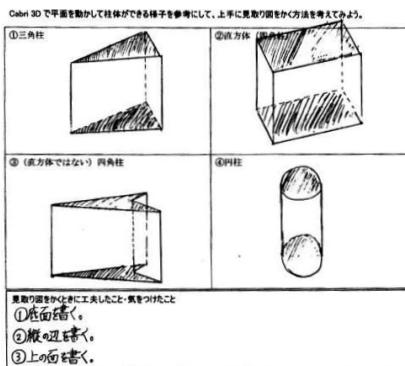
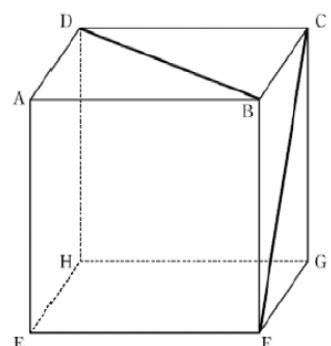


Figure 2: A student's description on the necessary ingenuity or precautions taken when they drew a sketch of prisms and cylinders (Sixth lesson)

“Draw the base plane, draw a vertical side, and draw the top plane”

Representation of a Spatial Figure on the Plane

Survey Question 3 (2001: No. 1B12, or 2003: No. 1B12) “In the cube described in the sketch on the right, we are going to compare the length of two line segments indicated with bold lines (—). Select the correct answers from (a), (b), or (c) below. Also, explain the reasons why in the column □. (a) Line



segment BD is the longest; (b) Line segment CF is the longest; (c) The length of line segments BD and CF are equal”

Table 2: The actual length of the line segment in the sketch with different groups

Group	Pass	Fail	Total
Classes in a same School			
With 3D DGS (n=66)	83.3%	16.7%	100
1998CS Compliant (n=32)	62.5%	37.5%	100
National survey 2003 (n=6822)	44.2%	55.8%	100

Table 2 shows results of the **Survey Question 3**. The pass percentage of the “With 3D DGS” (83.3%) was 20 points or more higher than that of “1998CS compliant” (62.5%). The Pass/Fail number of different groups regarding this question was significant, $\chi^2 (1, N=98) = 5.208, p < .05$. Similarly, the pass percentage of “With 3D DGS” was about 40 points higher than the percentage of the national survey 2003 (44.2%), and the Pass/Fail number was significant, $\chi^2 (1, N=6888) = 40.464, p < .01$. Therefore, it can be seen that the curriculum with 3D DGS heightens the pass percentage regarding this question comparing to that been compliant with 1998CS.

If attention is focused on the response status to the answer, the percentages of students who can understand that the diagonal line lengths are equal (answer type 5, 6, and 7) was 84.8% (n=66) in the “With 3D DGS” and 22.2 points higher than the one in the national survey 2003 (62.6%, n=6822). The relevant/not-relevant number regarding this answer type was significant, $\chi^2 (1, N=6888) = 13.862, p < .01$.

If attention is focused on relationship between cognitive and affective aspects, the percentage of students who passed and positively answered to the question “Did you understand the meaning of cutting a spatial figure with a plane and did you think about the shape of the cut surface?” was 81.8% (n=66) in the “With 3D DGS”. On the other hand, in the “1998CS compliant” it was 40.6% (n=32). Similarly, the percentage of students who passed and positively answered the question “Did you like studying cutting a spatial figure with a plane and thinking about the shape of the cut surface?” was 77.3% (n=66) in the “With 3D DGS”. On the other hand, in the “1998CS compliant” it was 38.8% (n=32). In addition, the percentage of students who passed and positively answered the question “Do you think that cutting a spatial figure with a plane and thinking about the shape of the cut surface will be useful in daily life or in the business world?” was 63.6% (n=66) in the “With 3D DGS”. On the other hand, in the “1998CS compliant” it was 24.8% (n=32).

In the “SPATIAL GEOMETRY” curriculum, 3D DGS induced the elaborateness and accuracy of observations and manipulating spatial figures. Therefore, it seems reasonable to conclude that students explored spatial figures based on their characteristics and relationships were enriched and the quality of findings and verifications were improved. For example, the students used 3D DGS to multilaterally observe the changes in the shape of cut surfaces (ex. a equilateral triangle, a square, a regular hexagon) by continuously moving the plane which cut a cube (9th & 10th

lessons: on Feb 8 & 9). In addition, in this process students could observe that it is not actually a regular pentagon when they operated and observed it from various angles even if the shape of the cutting surface looks like a regular pentagon from an angle.

Moreover, through activities related to the concept formation of spatial figures in described previously the fifth and seventh lessons, students cultivated their ability to understand or draw sketches. Therefore, it can be considered that the students could recognize that planes of a cube shown in the sketch were congruent. Moreover, as the activities relate to logical thinking, the students discovered and explained the characteristics of developments of a cube comparing developments which become a cube to those which do not become a cube (Second & Third lessons: on Jan 24 & 26). And in the discussion of the characteristics of a development which will become a cube or the discussion of the reasons that the cut surfaces of a cube do not become a right pentagon (11th lesson: on Feb 15), the activities included representing what they discussed and thought.

CONCLUSION AND IMPLICATIONS

The effects of the spatial geometry curriculum utilizing 3D DGS were found by comparing the result of experimental group with that of control group in the same school in Japan, as well as results of the national survey of Japan. In the result, as cognitive effects of the “SPATIAL GEOMETRY” curriculum, positive effects were identified regarding the construction of spatial figures by moving a plane figure, and the explanation for a spatial figure represented on a plane. Moreover, multiplier effects were recognized in the relationship between cognitive and affective aspects based on the students’ reflections on their learning about the construction and cutting of spatial figures respectively from three points of view (“understanding”, “interest level”, and “usability”).

The following two points could be considered as factors which led to those effects:

- In the “SPATIAL GEOMETRY” curriculum, the following qualitatively different activities were embedded respectively: Activities with concept formation of spatial figures, logical thinking, and representations of spatial figures
- The use of 3D DGS will facilitate the three possibilities in the above mentioned activities as follows: The possibility of exploring spatial figures those are physically difficult to construct or operate, multilaterally observing the process of the construction or operation of 3D figures in the dragging, and cultivating logical thinking about spatial geometry with dynamic transformations and multilateral observations.

As future research, the following challenges could be suggested: (a) In the learning of spatial geometry, to confirm effects of aspects not covered in this research. (b) To concretely specify factors that improved students’ learning to improve the curriculum as well as to compensate for the tendency captured by the quantitative analysis with qualitative analysis.

Note

- 1) This website was translated into English (<http://www.criced.tsukuba.ac.jp/gc/>).
- 2) The term “curriculum” is used not in the sense of a national Japanese curriculum in but rather in the sense of “a curriculum at the classroom level”. For each lesson plan, please refer to “<http://www.schoolmath3d.org/>”.

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MATHEMATICS AS MOTHER/BASIS OF SCIENCE IN AFFECT: ANALYSIS OF TIMSS 2003 DATA

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Mathematics is the mother/basis of science as revealed by the structure of our brain and the design of school subjects for most cultures. Does our motivational structure reflect this structure and in turn influence students' achievement? The results from a series of LISREL, correlation, and cluster analyses on TIMSS 2003 data showed that the pattern of roles of mathematics motivations in mathematics achievement is very similar to that in science achievement. However, science motivations play comparatively different roles between mathematics and science achievement. The results lend support to the claim that we establish the structure of our affective world as a reflection of the structure of our knowledge of the world and that mathematics is the mother/basis of science in terms of affect.

INTRODUCTION

Mathematics is the science of pattern and logic and can be a powerful tool to model a wide range of domains of world knowledge (e.g., science) and mathematics becomes an essential school subject, as revealed by the national curricula of most cultures. Mathematics or quantity is also one indispensable part in most IQ tests, e.g., WISC for both children and adults. Research on cognitive neuroscience also identifies a biologically determined part in human brain for the domain of quantity (Dehaene et al., 2003). Mathematics is therefore widely perceived as the mother/basis of science. Will our affective or motivational responses show a reflection of this knowledge structure? The exploration of cross-domain motivations in relation to cross-domain achievements can deepen our understanding of the structure of affect in learning mathematics and other subjects, e.g., science. Using student responses to the TIMSS 2003 study, we can examine this issue from a perspective of cross-cultural commonalities and diversity (e.g., Chiu, 2006). These examinations are likely to facilitate a sound design of teaching materials and classroom dialogue that takes account of the cognitive and affective structures of the learning of different domains of knowledge.

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Educators of different school subjects have long examined motivational/affective issues for their respective domains, e.g., McLeod (1994) and Hannula (2002) for mathematics and Tuan et al. (2005) for science, because these motivational constructs are significant predictors of achievement. The most significant construct is self-efficacy, which is the best affective predictor of student achievement especially when it is defined for specific tasks or domains of knowledge (Bandura, 1997; Pietsch, Walker, & Chapman, 2003). Wigfield and Eccles (2000) propose an expectancy-value theory of motivation, which comprises two factors: expectancy (i.e., ability/confidence and expectancy beliefs) and value (i.e., attainment importance, intrinsic value, utility value, and cost) and these distinct constructs are also evidenced domain-specific. Further, self-concept or these ability-related beliefs are viewed as a hierarchical or multidimensional construct (Marsh & Hau, 2004), i.e., the construct on the general level and the domain-specific level. There is however a limited understanding of the relationship between different domains of affect in relation to achievement.

In sum, according to the essence of mathematics and the design of our brain, IQ tests, and most national curricula, mathematics is the mother/basis of science and science includes more diverse domains of knowledge. If our motivational/affective system is a reflection of this design, we can expect to see that mathematics motivation is not only the basis of mathematics achievement but also science achievement, while science motivation will play a less consistent role in mathematics and science achievements. There may be some minor inconsistent findings for different cultures as some cultures may place little emphasis on the belief that 'mathematics is the mother of science' and may not design their school system accordingly.

METHOD

Participants

Forty-seven countries participated in the TIMSS 2003 study. However, the data for only 87,913 students from 19 countries (Table 1) were analyzed after excluding countries without motivation variables and without fit to the analytical procedure of LISREL (e.g., listwise deletion and non-definable parameters).

Indicators

Four kinds of indicators were taken from the TIMSS 2003 study:

(1) *Mathematics motivations*, referring to students' motivations about learning mathematics. The first part is *self-confidence* in learning mathematics (4 items, labeled *mc1-mc4* in the present study). The items are:

I usually do well in math. (*mc1*; TIMSS-variable BSBMTWEL).

Mathematics is more difficult for me than for many of my classmates (reversed, *mc2*; TIMSS-variable BSBMTCLM).

Mathematics is not one of my strength (*mc3*, TIMSS-variable BSBMTSTR).

I learn things quickly in mathematics (*mc4*, TIMSS-variable BSBMTQKY).

The second part is students' *valuing* mathematics (7 items, labeled *mv1-mv7*). The items are:

I think learning mathematics will help me in my daily life (*mv1*, TIMSS-variable BSBMAHDL).

I need mathematics to learn other school subjects (*mv2*, TIMSS-variable BSBMAOSS).

I need to do well in mathematics to get into the university of my choice (*mv3*, TIMSS-variable BSBMAUNI).

I would like a job that involved using mathematics (*mv4*, TIMSS-variable BSBMAJOB).

I need to do well in mathematics to get the job I want (*mv5*, TIMSS-variable BSBMAGET).

I would like to take more mathematics in school (*mv6*, TIMSS-variable BSBMTMOR).

I enjoy learning math (*mv7*, TIMSS-variable BSBMTENJ).

All the items use a 4-point rating scale ranging from 1 (*agree a lot*) to 4 (*disagree a lot*).

(2) *Science motivations*, including *confidence* (4 items, labeled *sc1-sc4*) and *value* (7 items, labeled *sv1-sv7*), with the same item content and scaling method as for those mathematics motivations, except for 'science' as the school subject.

(3) *Mathematics achievement*, including achievements in algebra, data, fractions/numbers, geometry, and measurement.

(4) *Science achievement*, including achievement in earth science, life science, physics, chemistry and environment/resources.

RESULTS

Measurement model

LISREL analysis reveals that a 3-factor model of mathematics motivation and science motivation is more acceptable than a 2-factor model for all 19 respective countries after a deletion and re-combination of the 11 items. The three factors are *confidence* (3 items), *utility* (2 items), and *interest* (2 items) for mathematics and science respectively (Fig. 1). The 3-factor model also reveals a better fit (RMSEA = .084) than the 2-factor model (RMSEA = .093) in the multi-group invariant tests using LISREL. The measurement model for the mathematics and science achievement reveals a slight fit (RMSEA = .109; CFI = .97) in the multi-group invariant tests.

Path model

A structural equation model of the role of mathematics/science motivation in mathematics/science achievement (Fig. 1) was proposed in order to examine the data from the 19 countries, respectively. This model is fit to all the 19 countries as revealed by the values of RMSEA (Table 1).

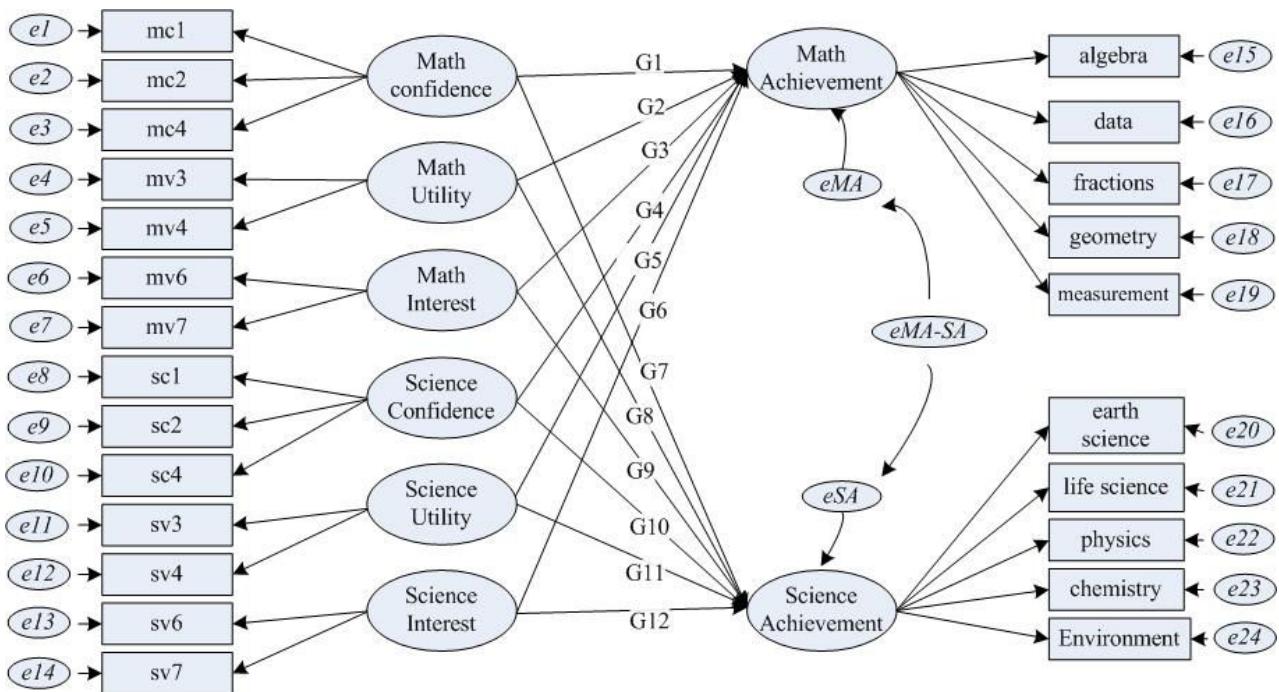


Figure 1: A structural equation model of mathematics/science motivation and achievement

Patterns of the roles

The degrees of significance of the path parameters (G1-G12) for the 19 countries were indicated by 3 as positive significance, 2 as non-significance, and 1 as negative significance. Bivariate correlations between the pairs of significance of G1 vs. G7, G2 vs. G8, G3 vs. G9, G4 vs. G10, G5 vs. G11, and G6 vs. G12 were .85, .83, .78, .38, .71, and .61, all significant at .01 level except for that of G4 vs. G10 ($r = .38$). These correlations reveal that mathematics motivations play similar roles in both mathematics and science achievement, while the roles of science motivations in mathematics and science achievement are quite different, especially for confidence in science.

Cluster analysis was utilized to identify the non-linear relationship between the degrees of significance of the path parameters of G1-G12. This analysis categorized the 19 countries into 3 clusters. One-sample T tests were performed to determine whether the degrees of significance of G1-G12 were significant from 'non-significance' (i.e., '2'). Table 2 shows that motivations in mathematics play the same pattern of roles in both mathematics and science achievement for the three clusters, while motivations in science are less stable in relation to both mathematics

Country	G1	G2	G3	G4	G5	G6	G7	G8	G9	G10	G11	G12	RMS-C	EA	M	DCC
New Zealand	-1.02*	-.10	.70*	.42*	.07	-.56*	-.71*	-.17	.76*	.06	.05	-.45	.068	1	.85	
Chile	-.95*	-.03	.71*	.24*	.00	-.19	-.73*	.01	.61*	.01	.01	-.11	.059	1	1.10	
Australia	-.79*	-.07	.42*	.33*	-.02	-.42*	-.49*	.04	.29*	-.05	-.19*	-.11	.065	1	1.49	
Norway	-.93*	.10*	.25*	.27*	-.04	-.23*	-.65*	.00	.34*	-.03	-.05	-.21*	.061	1	1.57	
Morocco	-.73*	-.11	.53*	.27	.44*	-.63*	-.16	.15	.03	-.27	.17*	-.07	.060	1	1.65	
Botswana	-.66*	.11	.45	.61	.21	-1.18*	.03	.01	-.02	.04	.00	-.46*	.064	1	1.72	
Japan	-.80*	-.18*	.40*	.25*	.08	-.36*	-.18*	-.38*	.18*	-.37*	.23*	-.05	.071	1	1.79	
South Africa	-.46	-1.50*	1.55*	.22	1.12*	-.91	-.31	-1.61*	1.61*	.24	1.25*	-1.16	.057	1	2.17	
England	-.49*	-.04	.24*	-.09	-.07	.04	-.31*	-.06	.33*	-.31*	-.10	.04	.063	2	1.27	
Italy	-.79*	-.06	.38*	.05	-.04	-.06	-.51*	.08	.24	-.2	-.18*	.13	.061	2	1.42	
Korea	-.82*	.21	.07	.14*	-.3*	-.04	-.33*	-.17	.17	-.34*	-.01	.00	.059	2	1.56	
Jordan	.26	-1.12	.52	-.96*	.63	.41	.20	-.98	.61	-.92*	.68	.16	.060	2	1.62	
Syrian Palestini an	-.52	.12	.11	-.51*	-.06	.55	.08	-.18	-.04	-.68*	.00	.56	.051	2	1.62	
United States	-.27*	.00	-.04	-.57*	-.15	.55*	-.2*	5	-.05	.18	-.61*	-.18	.45*	.057	2	1.68
Iran	-.69*	.20*	.18	-.22	.07	.13	-.42*	.02	.30*	-.33*	.15*	.03	.057	2	1.79	
Bahrain	-.55*	.36*	-.20	-.47	-.63*	1.03*	-.64*	.51*	-.01	-.24	-.45*	.48	.064	2	1.85	
Chinese Taipei	-1.77*	1.51*	.04	1.24*	-1.24*	-.41	-1.21*	1.32*	-.17	.66	-1.08*	-.14	.074	2	2.05	
Philippines	.39*	-.08	-.54*	.02	-.26*	.21	.33*	.09	-.38*	.23*	-.41*	-.13	.060	3	.00	

G1: math confidence → math ach; G2: math utility → math ach; G3: math interest → math ach; G4: science confidence → math ach; G5: science utility → math ach; G6: science interest → math ach; G7: math confidence → science ach; G8: math utility → science ach; G9: math interest → science ach; G10: science confidence → science ach; G11: science utility → science ach; G12: science interest → science ach; RMSEA = Root Mean Square Error of Approximation (< .08, as reasonable fit in LISREL); CM=cluster membership; DCC=distance from cluster center.

* Significant at .05 level

Table 1: Achievements by test results using LISREL, cluster, and K-means analysis

and science achievement. For Cluster 1 and Cluster 2, confidence in mathematics play a positive role and utility of mathematics failed to play a significant role in either mathematics or science achievement. Interest in mathematics has a negative relationship with both mathematics and science achievement for Cluster 1, but for Cluster 2 no significant relationship. For Cluster 1, science confidence has a negative relationship with mathematics achievement, but no relationship with science achievement. For Cluster 2, science confidence has a positive relationship with science achievement, but no significant relationship with mathematics achievement. For Cluster 1, science utility failed to predict both mathematics and science achievement, but for Cluster 2 science utility has a positive relationship with mathematics achievement but no relationship with science achievement. For Cluster 1, science interest has a positive relationship with mathematics achievement, but no significant relationship with science achievement. For Cluster 2, science interest has

no significant relationship with either mathematics or science achievement. In summary, for Cluster 1, science confidence is negatively and science interest is positively related to mathematics achievement, but not to science achievement. For Cluster 2, science confidence is positively related to science achievement, and science utility is related to mathematics achievement. For Clusters 1 and 2, mathematics motivations make more significant predictions (6 cases) than science motivations (4 cases).

K-means analysis obtained the distances from the respective cluster center for each of the countries (the last two columns in Table 1) and reveals that New Zealand is the most typical amongst the countries in Cluster 1, England is the most typical of those in Cluster 2, and the Philippines is the only case in Cluster 3. The significance pattern of G1-G12 for New Zealand is completely the same as that of Cluster 1, as revealed in Table 2, while the pattern of G1-G12 for England is slightly different from that of Cluster 2. The results for the Philippines also show that mathematics motivations play the same roles in both mathematics and science achievement. However, while the results for Clusters 1-2 show that mathematics confidence plays a positive role and mathematics interest negative or neutral, the Philippines is a mirror image, with mathematics confidence play a negative role and mathematics interest positive. There is the same pattern of relationship between science motivations and mathematics achievement for both Cluster 2 countries and the Philippines, with a positive relationship between science utility and mathematics achievement and no significant relationship between science confidence/interest and mathematics achievement. In the Philippines, science utility is also positively related to science achievement, but science confidence negatively.

	G1	G2	G3	G4	G5	G6	G7	G8	G9	G10	G11	G12	
Cluster 1	Mean	1.13	1.88	2.88	2.63	2.25	1.25	1.38	1.75	2.75	1.88	2.25	1.75
	SD	.35	.64	.35	.52	.46	.46	.52	.46	.46	.35	.71	.46
	T	-7.00*	-.55	7.00*	3.42*	1.53	-4.58*	-3.42*	-1.53	4.58*	-1.00	1.00	-1.53
Cluster 2	Mean	1.20	2.30	2.30	1.80	1.60	2.20	1.20	2.20	2.30	1.30	1.70	2.10
	SD	.42	.48	.48	.79	.52	.42	.42	.42	.48	.48	.67	.32
	T	-6.00*	1.96	1.96	-.80	-2.45*	1.50	-6.00*	1.50	1.96	-4.58*	-1.41	1.00
Cluster 3: Philippines		.39*	-.08	-.54*	.02	-.26*	.21	.33*	.09	-.38*	.23*	-.41*	-.13

* Significant at .05 level

Numbers in gray indicate the same patterns of significance in the role of mathematics (or science) motivations in both mathematics and science achievement, i.e., comparing G1 vs. G7, G2 vs. G8, G3 vs. G9, G4 vs. G10, G5 vs. G11, and G6 vs. G12

Table 2: One-sample T test results for the degrees of significance in path parameters of Clusters 1 and 2, cf. Cluster 3

DISCUSSION

As human beings, we all have some shared meanings, commonalities, and understanding of 'truth', while diversity makes different people and cultures unique. The present study identifies three commonalities that are shared by people of different cultures. First, using a series of LISREL analysis, a three-factor model of mathematics/science motivations was confirmed. The three factors are confidence, utility, and interest, which is a combination of theories of expectancy-value (Wigfield & Eccles, 2000) and intrinsic-extrinsic motivations (Ryan & Deci, 2000). Second, the role of the three-factor model of mathematics/science motivations in mathematics/science achievement was confirmed by LISREL analyses for the 19 countries given the TIMSS 2003 database. Third, correlation tests for the degrees of significance of roles of mathematics/science motivations in mathematics/science achievement reveal that the pattern of roles of mathematics motivations in mathematics achievement is very similar to that in science achievement. However, science motivations, especially science confidence, play comparatively different roles between mathematics and science achievement. Based on the above three commonalities, a claim that may be posited is that mathematics is the mother/basis of science in terms of motivations/affects. We can use mathematics motivations to predict not only mathematics achievement but also science achievement, but we cannot use science motivations in the same way.

There is diversity between cultures in their patterns of roles of mathematics/science motivations in mathematics/science achievements, which may address the issue of the influence of experience derived from specific cultures. Each culture is unique in its patterns and each of the patterns should be explained based on the uniqueness of each culture. Cluster analysis is only a method to find cultures of similar patterns of roles for mathematics/science motivations in mathematics/science achievements. Except for the Philippines, the results for most countries reveal that mathematics confidence plays a positive role in both mathematics and science achievements and that mathematics utility is not significantly related to either mathematics or science achievements. Based on this finding, a more precise, supplementary claim that may be posited is that mathematics confidence is the mother/basis of both mathematics and science achievements. This claim broadens our understanding of domain-specific self-efficacy (Bandura, 1997): We establish the structure of our confidence world as a reflection of the structure of our world knowledge. Mathematics is the mother/basis of science not only in the cognitive aspect but also in the affective aspect. For educational practice, we need to introduce to students a positive affect set not only for the domain (e.g., science) but also for its basic, essential domain-related subjects (e.g., mathematics).

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MEDIATING MODEL BETWEEN LOGO AND DGS FOR PLANAR CURVES

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Recent educational studies in planar curves tend to approach with tools, such as Logo and DGS, which emphasize action perspective and relation perspective, respectively. In this article, we consider the concept of vector as a powerful idea for integrating both action and relation perspectives. Also we discuss a mediating model connecting action and relation perspectives by designing an integrated microworld environment, and its implication in mathematics education.

INTRODUCTION

Recent development in science and technology has made the validity of information shorter. Knowing how to create information came to take priority over owning a set of knowledge. Accordingly, knowledge creator, not a knowledge consumer is needed now. This means that learners are asked to take an active role of exploring the meaning of self through experiences. Recent learning environment emphasizes on activities such as conjecture, experiment, and observation through which learners can construct one's own language on a mathematical object, rather than focusing them on a set of given definitions.

Development in science and technology has also made it possible to create teaching and learning environment never witnessed before. Especially, new technology can provide us with dynamic and manipulative experimental environment with regard to rate of change such as graph and movement. In the same vein, recent studies on teaching and learning planar curves tend to explore its qualitative characteristics while traditional research had focused on the concept of correspondence in relation to functional thought. The qualitative calculus (Stroup, 2002) approaches the understanding of function not as correspondence, which is the key to defining what function is in modern mathematics, but as covariation, which is more intuitive. The Computer-based Ranger (Berry et al., 2003), Motion Detector (Nemirovsky et al., 1998) and MathWorlds (Kaput, 1998) are the tools for the qualitative approach to function and its graph.

This line of research has been done frequently in the environment like Logo, putting importance on procedural, active command, as well as in DGS (Dynamic Geometry Software) with emphasis on manipulative, relative command. Eisenberg (1995) argues that it is necessary to integrate these two environments. Abelson & diSessa (1980) also discusses the advantages of translation between two environments. We further argue

that a mediating model be established to enhance translation between two environments in one integrated environment.

This article aims to review planar curves from the perspectives of relation and action and looks into characteristics of environments based on each perspective. It also proposes an environment integrating two models while exemplifying one model case interconnecting them. Finally, it discusses mathematical implications of realizing the integrated model in a microworld.

THEORETIC BACKGROUND AND MICROWORLDS

Constructionism is both a theory of learning and a strategy for education. It builds on the “constructivist” theories of Jean Piaget, asserting that knowledge is not simply transmitted from teacher to student, but actively constructed by the mind of the learner. Children don't get ideas; they make ideas. Moreover, constructionism suggests that learners are particularly likely to make new ideas when they are actively engaged in making some type of external artefact -be it a robot, a poem, a sand castle, or a computer program- which they can reflect upon and share with others. Thus, constructionism involves two intertwined types of construction: the construction of knowledge in the context of building personally-meaningful artifacts (Kafai & Resnick, 1996, p. 1).

Computers and mathematics education (Cho, 2004), a field of study connecting mathematics education with computer, is theoretically based on constructionism which places stress on mental construction of concepts and knowledge through physical construction. Naturally, education based on constructionism needs a space for physical construction for its effective implementation. A microworld realizes this space in a computer. It emphasizes on learning rather than teaching, structure of the environment for knowledge construction rather than functional features of software for improving the effectiveness of knowledge delivery. This structure is closely related to ‘powerful ideas’, suggested by Bers (2001). He states that a microworld is a constructivist environment designed to generate powerful ideas in a space like a black-box.

Papert (1980) discusses the need of powerful ideas as a mediating thought while giving ‘feedback’ or ‘ideas on mediating cases’ as examples of powerful ideas. In designing a computer-based environment for constructivist learning, it is necessary to consider what tools and ideas should be given to learners in order to enhance their meaningful construction of knowledge. This article aims to suggest different presentations and further ‘composition and decomposition of vector’. Abelson & diSessa (1999) talks about advantages of employing different perspectives on a phenomenon. Freudenthal (1993), while describing the characteristics of vector, emphasizes its mathematical meaning.

Research has been conducted on various ways to encourage learners to engage in the constructivist learning with their sense of planar curves rather than focusing on correspondence, sets, calculation of them. What we particularly take note of is the research on planar curves in the Logo and DGS environment. Also, there have been

studies on how to approach planar curves using a turtle's movement. Armon (1999) and Kynigos (2000) explore the curves using intrinsic equation¹. Specifically, Armon (1999) attempts to find intrinsic procedure through intrinsic equation and applies it to turtle-algorithm. The essence of these studies lies in the transformation of intrinsic equation, to which turtle-algorithm is applicable, to turtle-action. Eisenberg (2000) aim to give practical meaning to curves like cycloid and cardioid by offering a command representing two actions of a turtle in one consequence.

Cha & Noss (2002) explains the importance of DGS design with reference to locus problems in functions and graphs. Maor (2003) states that many tools have been developed to draw different types of curves and that it is a great pleasure to see those tools move, slowly drawing designated curves. Spirographs, which gained popularity in 1970s, had been used as a tool to draw beautiful figures with periodicity and studied in mathematics, especially in relation to astronomy (Ippolito, 1999; Adams et al., 2006). This shows the importance of manipulative environment in dealing with curves.

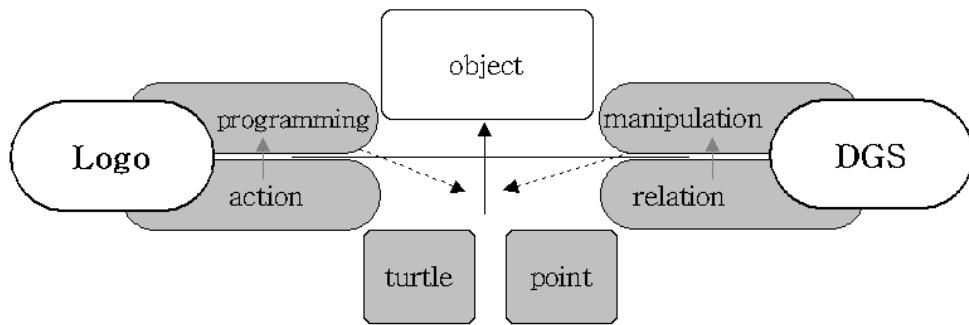


Figure 1: Logo and DGS

Logo and DGS are not totally distinct from each other. Rather, these two different environments can be integrated into one; Cho (2006), Cho et al. (2004)². Figure 1 represents how a set of actions and relations, central concepts in DGS and Logo, can work in an integrative way.

MEDIATING MODEL: ELLIPSE

JavaMAL is a microworld integrating Logo and DGS. A turtle's actions in Logo influence basic objects in DGS like point and line and vice versa. Integration of Logo and DGS is necessary in that it emphasizes on both construction and manipulation and their interaction. Sherin (2002) attempts to realize DGS from the perspective of turtle microworld language. Abelson & diSessa (1980) also discuss the translation between two different representations like the following:

Turtle geometry and vector geometry are two different representations for geometric phenomena, and whenever we have two different representations of the same thing we can

¹ Intrinsic equation, also called natural equation, refers to a formula represented by three variables of arc length, radius of curvature, and tangential angle.

² This refers to an environment in which learners can observe how the turtle moves by manipulating its speed bar, and can explore the possibility of gifted education by using turtle action to create, manipulate, and move diverse tiles.

learn a great deal by comparing representations and translating descriptions from one representation into the other. Shifting descriptions back and forth between representations can often lead to insights that are not inherent in either of the representations alone (Abelson & diSessa, 1980, p. 105).

We understood the concept of curve as (1) static one based on correspondence/set or (2) dynamic one based on parametric variables. However, it is difficult to relate, translate between, and capture the meaning of two representations. To facilitate the actions, we need a bridge between two representations. In this article, we call the bridge a mediating model.

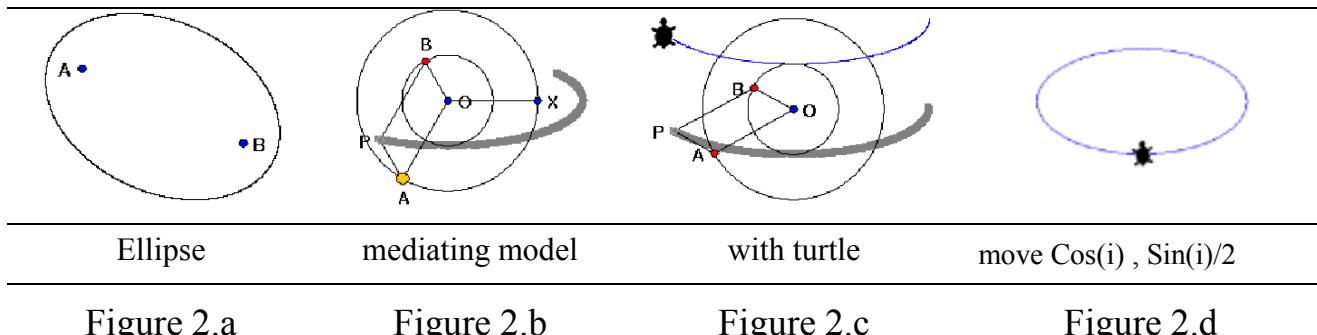


Figure 2: ellipse with DGS and Logo³

Figure 2.a and Figure 2.d are DGS ellipses created by ellipse create command and turtle action “move⁴”. Figure 2.b, drawn by trace of P when A is manipulated, is the mediating model, which contains both of the characteristics of DGS and Logo ellipse. In this article, we define a mediating model as follows.

There are two concentric circles O_1, O_2 , each of which radius is the arbitrary positive real numbers r_1, r_2 . Here, points A, B exist on circle O_1, O_2 , respectively and for arbitrary integer number n , $\angle AOX = n \cdot \angle BOX$ and $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$.

As in Figure 2.b, the point B moves on the smaller circle in the opposite direction of the point A moving on the bigger circle if $r_1 = 40$, $r_2 = 20$, $n = -1$. While A turns one time, so does B. Here the trace of B makes an ellipse.

This model is executed in the DGS environment. Where does a turtle hide? We can give a command which makes a turtle create the same circle in the above picture, like Figure 2.c. To make this possible, we have to calculate the distance the turtle covered and do the mathematical activity like seeking the relation formula between the distance and the perimeter of circles.

The movement of a turtle and the trace of P, which are represented by the relationship between two circle-making points, are based on the same principle. It is the composition of vectors, one of powerful ideas.

³ <http://www.javamath.com/class.cgi?343>

⁴ For more discussion on the move command, see Cho et al. (2004).

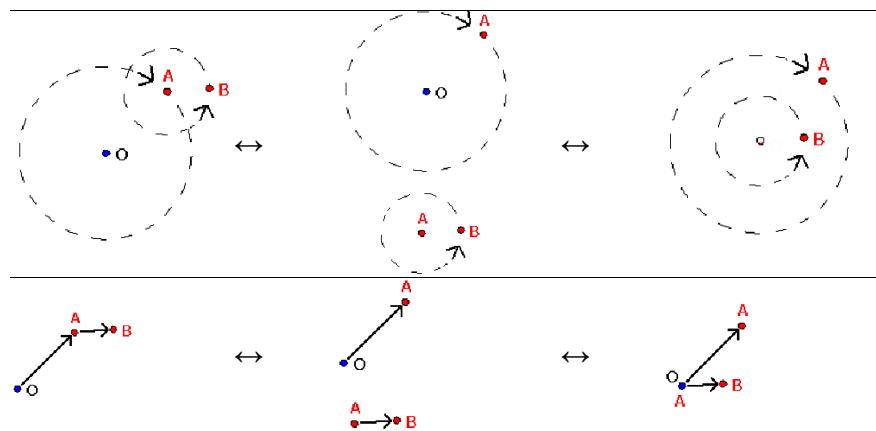


Figure 3.a

Figure 3.b

Figure 3.c

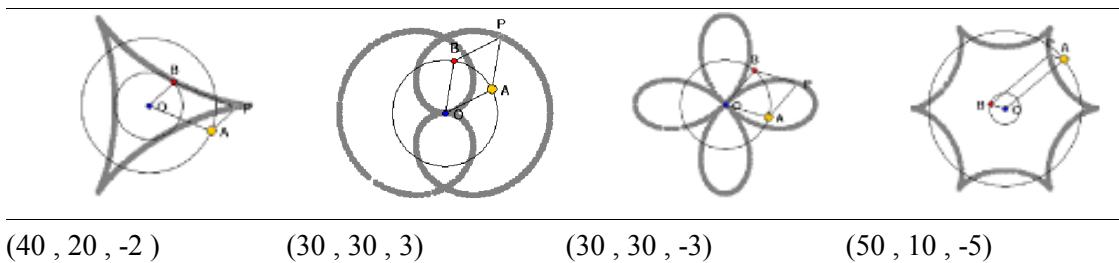
Figure 3: vector approach for ellipse

Figure 3.a represents the point A turning clockwise around the circular center O and the point B turning anti-clockwise as A's satellite. At a given moment, A's and B's revolving movement influence the satellite B, moving point B in the direction of two movements' sum. Using a powerful idea of sum of vectors, Figure 3.a can be translated into equivalent relations like Figure 3.b and Figure 3.c. Figure 3.c is an ellipse mediating model in this case. This line of thoughts can be represented briefly using the sum of vectors as follows.

$$\begin{aligned} \overrightarrow{OA} &= \overrightarrow{AB} + \overrightarrow{OB} \dots \textcircled{1} \\ \overrightarrow{OA} + \Delta \overrightarrow{OA} + \overrightarrow{AB} + \Delta \overrightarrow{AB} &= \overrightarrow{OB} + \Delta \overrightarrow{OB} \dots \textcircled{2} \\ \Delta \overrightarrow{OA} + \Delta \overrightarrow{AB} &= \Delta \overrightarrow{OB} \dots \textcircled{3} \end{aligned}$$

Formula ① represent point B's change of location based on the manipulation of point A in DGS while formula ③ describes the instant action of B. Notably, formula ② uses expressions in DGS and Logo simultaneously. Specifically, it has its own meaning while providing meanings to both DGS and Logo. This is why we argue that this kind of mediating model can bridge DGS and Logo.

We can get various planar curves by adjusting the values of r_1, r_2, n . Figure 4 shows planar curves obtained by representing different values of r_1, r_2, n into corresponding (r_1, r_2, n) .

Figure 4: planar curves with the model⁵

⁵ Planar curves by mediating model. <http://www.javamath.com/class.cgi?344>

PLANAR CURVES

We have looked at the model, which constructs an ellipse with movements of two circles, and its relationship with DGS and Logo environments. Now we are to apply our mediating model to more diverse types of planar curve.

The study of epicycloids goes back to the Greeks, who used them to explain a puzzling celestial phenomenon: the occasional retrograde motion of the planets as viewed from the earth (Maor, 1998, p. 102).

Epicycloid refers to the trace curve based on the relation between two circles. Originally developed in the endeavour to explain the celestial movement in astronomy. One of the main issues in epicycloid is the type of curve determined by the ratio of two circles' radii. Figure 5 represents a cardioid, the typical example of epicycloid, in DGS, our mediating model, and Logo, in turn.

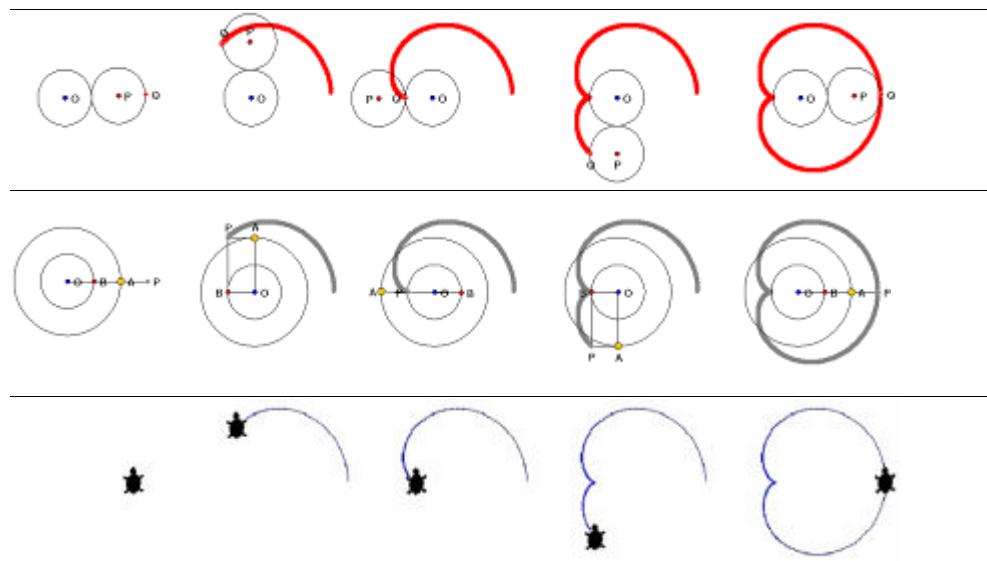


Figure 5: Cardioid with different representations⁶

Extension and generalization are important for students' creative and divergent thinking. There are different ways to secure extension and generalization. For example, we can use more than two circles, multiple points on each circle or increase the number of points, which determine an angular velocity. In the following we review the strategy to use multiple circles.

Points A, B, C, D are on four concentric circles with radius of 65, 20, 25, 45, respectively. Here,

$$\angle AOX = -2\angle BOX = -3\angle COX = 2\angle DOX \text{ and}$$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$. Given these conditions, the trace of P makes a fish curve like figure 6.

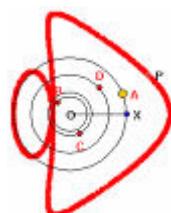


Figure 6: tropical fish⁷

⁶ <http://www.javamath.com/class.cgi?340>

⁷ Tropical fish-shaped curve using four circles . <http://www.javamath.com/class.cgi?341>

CLOSING REMARKS

In this article, one planar curve is drawn in a dynamic microworld and turtle microworld. It is also drawn using a ‘powerful idea’, which views the curve as sum of two vectors. Turtle Microworld offers the environment in which various curves are created, explored, and manipulated with turtle commands and composition / decomposition of vectors from the perspective of ‘what if’.

We can find the educational meaning of this environment in that it offers the venue in which various actions are created. Further, those created actions can give birth to manipulable objects like Tile. These features can foster learners’ creative thinking and explorative activities. Further, the integrated environment can help introduce the dynamic concept of planar curve and vector to mathematics education earlier and more naturally than now.

Specifically, the environment suggested in this article is distinct from traditional software, which focuses on delivery of information for the quantitative and numeric understanding of planar curve, in that it gives the possibility for learners to understand natural phenomena, functional graphs, and various curves in an idea and construct, and manipulate them in a microworld. It also emphasizes intrinsic and local procedure in approaching geometric phenomena, which may help learners grasping a central concept from different perspectives, explore the different aspects of variate in calculus, and understand trigonometric function more easily by using it as a tool.

Finally, this article reviews what is required to connect different mathematical ideas in the integrated environment and what examples can be utilized. It also deals with the issues in applying the integrated environment to education. This has significant implication in the current curriculum which places stresses on integrative thinking ability. Further research is needed on the meaning of this environment in mathematics curriculum so that it can help reorganize and enhance the subject matter.

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COMPARING KOREAN AND U.S. THIRD GRADE ELEMENTARY STUDENT CONCEPTUAL UNDERSTANDING OF BASIC MULTIPLICATION FACTS

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Elementary students who were currently third graders in Korea and the U.S.A. were asked to complete the Researcher-made Multiplication Questionnaire (RMQ) that contained four open-ended questions in order to examine their conceptual understanding of multiplication. The U.S. third graders provided better and more divergent definitions of multiplication than Korean students. However, Korean students created a multiplication word problem and represented it with a number sentence better than U.S. students did. Also, Korean students better identified the situations when multiplication was used in their real lives. But, Korean students lacked understanding of the different meanings of multiplication for all questions compared to the U.S. students' understanding.

INTRODUCTION

An agenda for the reform of school mathematics in the field of education has been the topic of considerable research, resulting in a new teaching approach based on the current learning theory "Constructivism." The teaching and learning of mathematics has moved worldwide from memorizing sets of established facts, skills, and algorithms to focusing on children's active construction of meaning and mathematics as sense-making and meaningful learning.

In the U.S.A., this movement has been articulated with consistency by publications of the National Council of Teachers of Mathematics (NCTM) such as the *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Principles and Standards for School Mathematics* (2000). These publications call for reform in school mathematics by placing emphasis on the importance for all students in grades K-12 to study a common core of broadly useful mathematics through their active participation in the learning process in order to become intellectually autonomous learners.

Korean schools use a national curriculum. This mathematics curriculum has been developed and revised by a committee of educational leaders among classroom teachers in different grade levels, mathematics educators, and researchers from academic institutes under the authorization of the Ministry of Education and Human Resources Development (MEHRD). The current Korean mathematics curriculum, which is the 7th national curriculum, distinguished as "learner-centered," was revised in 1998 (Lew, 2004) and has been implemented since 2000 (Paik, 2004, p. 12). This curriculum development reflected the current mathematics reform movement within

the international context and was easily noticed in the history of Korean mathematics education (Paik, 2004).

International comparative studies in student academic achievement, such as the Trends in International Mathematics and Science Study (TIMSS, 1999 & 2003) and the Program for International Student Assessment (PISA, 2003), reported that Asian students achieved high scores in the subject of mathematics and outperformed their western counterparts. Particularly in 2003, Korean 8th graders ranked 2nd (M=589) while the U.S. students ranked 15th (M=504) in mathematics among 46 countries participating in TIMSS. After these findings, there is a growing research interest in Asian mathematics education within the U.S.A. Various research studies have been carried out to contrast the curriculum and instructional methods to examine differences between Asian countries and western countries (e.g., Chung, 2005; Hiebert & Stigler, 2000; Li, 2000; Watanabe, 2001; Yong, 2005). However, few research studies contributed to comparing students' conceptual understanding of mathematics, which is one of the most important current issues in the teaching and learning of mathematics.

Multiplication has been much more difficult for children to perform than addition and subtraction in general (Oliver, 2005). Earlier studies have shown that young children can develop multiplication concepts in kindergarten or first grade (Carpenter, et al, 1993; Clark & Kamii, 1996). However, teaching multiplication facts is a basic part of the primary grade (K-3) mathematics curriculum. Students are introduced to multiplication concepts in second grade and are required to memorize their facts in third grade (Chung, 2005; Wallace & Gurganus, 2005). Some researchers (e.g., Behr et al., 1994; Bell et al., 1989; Confrey & Smith, 1995) claimed that once children reach the primary grades they are unable to solve problems involving multiplication or apply multiplicative number facts with meaning. When students reach grades 4-5, they experience difficulty in using multiplicative reasoning in a range of contexts and integrating their understanding of rational numbers with multiplication and division. This suggested that difficulties faced by older students can be attributed, at least in part, to the lack of development of conceptual understanding of multiplication in early primary grades (Mulligan & Watson, 1998). With the results of the international comparative studies and concerns about teaching and learning mathematical concepts, especially multiplication concepts in primary grades, researchers investigated how students from Korea and the U.S.A. differ in their conceptual understanding of basic multiplication facts.

Purpose

This study was established to compare the conceptual understanding of basic multiplication facts between Korean and U.S. 3rd grade elementary students. The specific objectives of this study were to investigate the following: 1) Define multiplication; 2) Represent their understanding of multiplication at the symbolic level (such as numbers and words, and number sentences); 3) Communicate their

understanding of basic multiplication facts with others; and 4) Relate their conceptual understanding of multiplication to their real life situations.

METHODOLOGY

Participants

Participants for this study included 129 male and 111 female, (total 240) third grade students of Korean (n=120) and American (n=120) heritage who were currently enrolled at two public schools in the suburban areas of the states of Indiana and Illinois in the U.S.A. and three schools in the suburban areas of Seoul and Pusan in Korea. The mean ages of the participants were 111.25 months (SD 3.48) for Korean schools and 106.35 months (SD 4.16) for the U.S. schools. Korean students were taught with the national elementary school mathematics curriculum which was developed by a mathematics curriculum committee under the authorization of the Ministry of Education and Human Resources Development (MEHRD, 2001). The U.S. students who participated in this study were taught with the mathematics curriculum, "*Everyday Mathematics*" published by the University of Chicago School Mathematics Project (2001).

Instrument

The Researcher-made Multiplication Questionnaire (RMQ), entitled "How Much Do I Know about Multiplication?" was used to examine the 3rd graders' conceptual understanding of basic multiplication facts. It consisted of 4 open-ended questions. Students were asked to define multiplication, create a multiplication word problem and construct a number sentence for the problem, explain their understanding of a basic fact (e.g. 7 x 6) using their own words, and describe how they utilize multiplication skills in their real life situations.

The questionnaire was developed in English and reviewed by two professors. The first is a university professor with expertise in educational measurement and statistics and the second is a mathematics education professor. The questionnaire was translated into Korean by the researcher. This instrument was also used in a previous research study done by Kim, Anderson, and Chung in 2002. For the current study, the RMQ was reviewed and revised by two classroom teachers from each country prior to distribution to the participants.

Procedure and Data Analysis

The researcher in each country distributed a letter containing the information explaining the objectives of the study, a copy of the parent consent form, student consent form, and the questionnaire to the principals of three participating schools at the beginning of the fall 2006 semester. One school in the U.S.A. dropped out from the study. As a result, only five public schools participated in this study. Two schools were located in Seoul and one in the Pusan area in Korea. One school was located in Chicago, Illinois and one school was located in Granger, Indiana in the U.S.A.

The student questionnaires were collected by the researchers in early September from Korean schools and during the period of mid-November to early December from the U.S. schools.

Student responses on the questionnaires were categorized following the guidelines of the data coding system developed by the researchers and input on the computer using SPSS 14.0 software. The four multiplicative structures (additive/equal group, array/area, multiplicative comparisons, and combinations) identified by Greer (1992) were adopted to analyse student responses regarding multiplication stories. The coding system and input data were cross-examined by both researchers and reviewed by a Korean American sociology professor who is an expert in research measurement and statistics. Coded data were analysed using descriptive statistics. Frequencies, percentages, and cross-tabulation were used to analyse student responses to individual items on the RMQ. Cross-tabulation and the Chi-square statistics were employed to determine differences in the third grade students' conceptual understanding of multiplication between the two groups.

RESULTS

When students were requested to provide a definition of multiplication using their own words, approximately three quarters of the U.S. students ($n=88$, 73.3%) provided a correct definition whereas only about one half of the Korean students ($n=56$, 46.7%) could do so. The Pearson Chi-Square statistics indicated that more U.S. students had significantly clearer definitions of multiplication than Korean students at $p < .001$ (see Table 1).

There are four distinctive models in multiplication. These are additive/equal groups, area/arrays, multiplicative comparisons, and combinations/Cartesian. The additive model tells how many groups or sets of equal size are being considered. The area/array model is a rectangular region defined as the units along its length and width and an arrangement of objects or pictures in rows by column. The multiplicative comparisons mean that there are two different sets that needed to be matched one-to-one to decide how much larger one is than the other. The combination/Cartesian model states that there are two factors representing the sizes of two different sets and the product indicates how many different pairs of things can be formed (Reys et al., 2004).

Students from both groups provided definitions in five different ways: 1) Additive/equal groups; 2) area/array; 3) multiplicative comparisons; 4) additive and array; and 5) additive and multiplicative comparisons. For the latter two ways, students explained the problem using two different models. Approximately forty-two percent ($n=101$) of two hundred forty students (48 Korean students, 53 U.S. students) defined multiplication as additive/equal groups (see Table 2). For the task to create a multiplication word problem, about seventy-one percent of Korean students ($n=85$) and about fifty-six percent of U.S. students ($n=67$) provided correct word problems and represented the problems in a number sentence. Pearson Chi-Square showed significant statistical difference between the two groups at $p < 0.001$ for the word

problem and at $p < .01$ for representing the problem by a number sentence (see Table 1). Approximately fifty-five percent of students (76 Korean students, 55 U.S. students) created additive/equal group models of multiplication problems (see Table 2).

Students were requested to explain to their younger siblings what 7×6 means in their own words. For this question, approximately seventy-three percent of Korean students ($n=88$) and seventy-six percent of U.S. students ($n=91$) explained the meaning of 7×6 correctly. There was no statistical significant difference between the two groups at $p > .05$ for explaining the meaning of the problem (see Table 1). Approximately twenty-eight percent of U.S. students ($n=33$) explained the problem with words and about twenty-six percent of students ($n=31$) used numbers and words in their explanation. Close to seventy-five percent of Korean students ($n=85$) described the problem in words and numbers/number sentences. The most frequently used approach for Korean students ($n=81$, 67.5%) was to describe the meaning of the problem with the additive/equal group multiplication model. For the U.S. students, about forty-one percent of the students ($n=49$) approached the problem using additive/equal groups and twenty-three percent of the students ($n=28$) explained the problem using the array/area model of multiplication (see Table 2).

Regarding the question of when students use multiplication skills in their real lives, nearly one half of the Korean students ($n=56$, 46.7%) and slightly more than one forth of the U.S. students ($n=33$, 27.5%) clearly identified a situation. Pearson Chi-Square statistics revealed that there was significant difference between the two groups at $p < 0.01$ for identifying situations, but no differences in terms of relating how to use the skills in their real lives ($p > .05$) (see Table 1). Thirty-five percent of Korean students ($n=42$) and twenty percent of U.S. students said they had used multiplication skills as the additive/equal group model in their real lives (see Table 2).

Tables

* $p < .05$, ** $p < .01$, *** $p < .001$

	KOREA (n=120)			USA (n=120)		
	Correct	Incorrect	No Resp.	Correct	Incorrect	No Resp.
Definition of multiplication***	46.7%	52.5%	0.8%	73.3%	26.7%	0%
Creating a word problem***	70.8%	17.5%	11.7%	55.8%	43.3%	0.8%
Connecting a word problem to a number sentence **	70.8%	29.2%	0%	54.6%	45.4%	0%
Explaining the meaning of the problem	73.3%	25.0%	1.7%	75.8%	23.3%	0.8%
Identifying the situations**	46.7%	50.0%	3.3%	27.5%	69.2%	3.3%
Explaining real life applications	36.7%	60.0%	3.3%	25.0%	71.7%	3.3%

Table 1: Third graders' conceptual understanding of basic multiplication facts (N=240)

* $p < .05$, ** $p < .01$, *** $p < .001$

	Additive(A)	Array(R)	Multiplicative(M)	A & R	A & M	Incorrect	No Resp.
Definition of multiplication ***							
KOREA	40.0%	0.8%	5.8%	0%	0%	52.5%	0.8%
USA	44.2%	20.8%	4.2%	7.5%	0%	23.3%	0%

Creating a word problem							
KOREA	63.3%	4.2%	5.8%	0%	0%	25.0%	1.7%
USA	45.8%	5.8%	7.5%	0.8%	0%	39.2%	0.8%
Explaining the meaning of a problem***							
KOREA	67.5%	2.5 %	4.2%	0%	0.8%	23.3%	1.7%
USA	40.8%	23.3%	5.0%	11.7%	0%	18.3%	0.8%
Identifying the situations							
KOREA	35.0%	5.0 %	2.5%	0.8%	0%	53.3%	3.3%
USA	20.0%	5.8%	1.7%	0%	0%	69.2%	3.3%

Table 2: Models of multiplication situations (N=240:120 for each country)

DISCUSSION AND CONCLUSIONS

The analysed data of this study yielded results supporting previous international comparative studies. Korean third graders outperformed the U.S. students in creating multiplication word problems, constructing a number sentence to represent a word problem, and identifying real life situations in which they utilize multiplication skills. The *Principles and Standards for School Mathematics* (NCTM, 2000) advocates that a balance and connection between conceptual understanding and computational proficiency are required for students to develop fluency in mathematics. For example, when multiplication facts are taught for conceptual understanding and connected to other mathematics concepts and real-world meaning, students perform better on standardized tests and in more complex mathematics applications (Campbell & Robels, 1997). This study indicates that Korean students have good understanding of multiplication concepts.

One of the interesting findings in this study was that U.S. students gave better definitions of multiplication, yet they were not able to apply this knowledge to creating a multiplication word problem and a number sentence that corresponded with the problem. In the meantime, Korean students were better able to explain word problems and construct the number sentences. However, their ability to explain the meaning of the specific problem (7×6) was not significantly better than that of the U.S. students. Korean students better identified the situations in which they needed to use multiplication in their real lives, even though they were not more capable of explaining how multiplication skills were specifically used. This implies that Korean teachers focus more on having students practice constructing word problems and number sentences and understanding the word problems rather than letting students explore and connect relationships among symbolic representations and different concepts.

Another important finding in this study suggests that there is a major conceptual instructional challenge for classroom teachers of both groups. When teaching multiplication, teachers need to help children understand that multiplication has a variety of meanings, which are additive/equal group, array/area, multiplicative comparison, and combination/Cartesian. The models that were provided by third

graders in this study were not diverse. The greatest number of students in both countries used the additive/equal group model, which was defined by the students as repeated addition. Korean students, in particular, dominantly used the additive/equal group model. The array/area model was the second most frequently used model by the U.S. students to illustrate the multiplication problem. This indicates that the U.S. students possessed more various meanings of multiplication. Few students used multiplicative comparisons and no student provided multiplication meaning as the combination/Cartesian model.

Finally, one of the most significant findings in this study was that students from both countries did not clearly explain how multiplication skills were used in their real lives. More than three fifth of students (60.0% Korean students, 71.7% U.S. students) could not accurately address the question, which was to discuss how multiplication skills are used in everyday life by providing a specific example.

Wallace and Gurganus (2005) recommended that the most effective sequence of instruction to help children acquire the concepts of multiplication facts is to introduce the concepts through problem situations and link new concepts to prior knowledge. With this strategy, students also should be allowed to have concrete experiences and semi-concrete representations before purely symbolic notations, explicit instruction of rules, and mixed practice are introduced.

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SELF-MONITORING BY LESSON REPORTS FROM TEACHERS IN PROBLEM-SOLVING MATHS LESSONS

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The following article presents chosen results from a teacher training on the learning of mathematical problem-solving in connection with self-regulation. The lessons documented during several weeks and the work products submitted by the teachers show specific further training effects.

INTRODUCTION AND THEORETICAL BACKGROUND

In Germany education standards (KMK, 2003) have been developed which are implemented at the moment. Problem-solving plays a central role in these standards, and special competencies are required from the teachers to enhance problem-solving in maths lessons. Within the scope of a project supported by the DFG (German Research Foundation) the Technical University Darmstadt pursues the aim to enhance this teaching competency on the basis of an approved, material-based and daily-life adapted teaching concept for the learning of problem-solving in connection with self-regulation (Komorek et al., 2006). The results of a study with student training measures on specific problem-solving strategies and interdisciplinary self-regulation strategies reveal specific effects regarding the mathematical performance of the students (Perels et al., 2005). On the basis of these training results a teaching concept for the enhancement of problem-solving in connection with self-regulation was established for teacher further training courses (Collet et al., 2006). The teacher further training of the school year 2004/2005 was focussed on the following aspects:

- A teaching concept for the learning of problem-solving
- The enhancement of the self-regulation of students by homework

Teachers who participated in the study were trained on special research contents like problem-solving (PS), problem-solving and self-regulation (PSR), self-regulation (SR) and the safeguarding of mathematical basis competencies (CG: “Quasi-control group”¹). In a further training at the beginning of the school year and supported by supervision (curricular-based (CB), webbased (WB), no supervision (NO)) during the school year the teachers had to go through defined fields of competency, depending on the further training content. Following a (moderate) constructivist approach and in order to reach corresponding effects with the teachers, practical exercises were part of the further training at the beginning of the school year and of the supervision during the

¹ CG-group is no control group in the proper meaning of the word as the teachers of the group underwent further education for mathematical basic competencies and internal differentiation.

school year, allowing to experience problem-solving strategies or to construct own problems in the sense of the teaching concept.

Theoretical background and classification of the study

Shulman (1986) describes six knowledge categories of teacher competencies. The present study analyses the PCK (pedagogical content knowledge) in connection with action elements for problem-solving in combination with self-regulation. The problem with this kind of studies is that there is no clear horizon of expectation (standards) in Germany so far. In the American standards for maths teachers (NBPTS, 2001) those competencies are already included, however not operationalized. The purpose of the present study is to contribute to the development and the use of suitable instruments to describe further training effects. Qualitative instruments and conventional quantitative methods as well as cross-section and process surveys for the analysis of efficiency are applied in a field work with both teachers and students, especially lesson reports and work products. The work products are particular problems, long-term homework, learning controls and teaching drafts. On the basis of submitted work products it is possible to assess if a teacher is able to implement developed knowledge of subjects treated in his further training. The lesson reports of the teachers allow to evaluate their self-perceived ability with regard to the integration of the further training concept in regular maths lessons.

Questions and hypothesis concerning the chosen tools

The present study wants to analyse the following research questions:

- To what extent are self-developed tasks reflecting further training effects? (*work products*)
- Which effects on the self-perceived implementation of the further training content by the participating teachers has the intervention with the further training concept? (*lesson report*)

Positive trends concerning the implementation of the further training content are expected from the lesson reports for those further training groups which were supervised throughout the school year (CB, WB). The tasks developed by the teachers were expected to show the individual implementation of the concept, following Galperin (1974) in three grades of professionalism levels depending on different action orientations (try-and-error-, pattern-, field orientation). The aim of the further training is to acquire at least one *pattern orientation*. This means that only examples without variation are adopted. However, in the long run the aim is to achieve *field orientation*, allowing the teacher to generate own examples which are in line with the concept. The quality of the work products alone does not guarantee the successful implementation of the concept in maths lessons but is considered as an essential prerequisite.

STUDY DESIGN

48 teachers (Gymnasium teachers and teachers from other school types) from 9 schools with classes of levels 7 and 8 took part in the study. At the beginning of the school year 2004/2005 the teachers participated in a further training with four training modules according to the aforementioned competencies (PS, PSR, SR, CG). Two groups were supervised throughout the school year, either by curricular-based training courses (CB) or by webbased coaching (WB). Table 1 shows the design of the study with different variations of the further training content and coaching.

	PS	PSR	SR	CG
CB		4	8	
WB	11	8	4	
NO		6		8

Table 1: Design of further training method and content: number of teachers

Instruments of data collection

Four developed instruments were applied with the *teachers* to analyse the further training effects from different points of view (www.math-learning.com). A *repertory grid survey* (qualitative) and a *teacher questionnaire* (quantitative with qualitative elements) served as basis for a pre- and post comparison. The repertory grid survey allows to record the ideas teachers may have on maths problems by asking them to specify characteristic features of problem pairs (Collet et al., 2006; Lengnink et al., 2003). The teacher questionnaire deals with teaching aims in connection with the enhancement of subject-related and multidisciplinary competencies, attitude and experience with respect to the integration of individual learning possibilities in the lessons and cognitive requirements in the homework. Moreover the teachers' ideas about good maths lessons are collected in concept maps. For the internalization of aspects of the teaching concept and the documentation of the concept implementation a self-monitoring instrument, the *lesson report*, was used as processual investigation tool. The teachers were asked to document their maths lessons continuously over a period of 10 weeks. The lesson report consists of 34 items established against the background of the teaching concept on the learning of problem-solving and self-regulation in connection with basic structural elements of "good" maths lessons. In order to gain access to their knowledge regarding the successful concept implementation and to maintain the teacher further training effects by individual experience, the teachers are asked to submit at least one own *work product* at the end of the further training. Moreover a *student performance test with problem-solving tasks* was run to show possible effects of the further training (Collet et al., 2006). A *standardized student questionnaire* was employed to determine, among others, the self-regulative competency and the perception of the lessons. Both instruments were adopted in the pre- and post-survey.

RESULTS OF THE STUDY

Selected general results

The *repertory grid survey* revealed quantitative and qualitative growth in the description of problem features, allowing to draw corresponding conclusions from the further training effects (Komorek et al., 2006). The *teacher questionnaire* demonstrates the stability of the values allocated to subject-related and multidisciplinary teaching aims as well as to the commitment and self-perception of teachers. There is a significant increase in the enhancement of individual learning and cognitive homework requirements. The students manifested in their *student performance test* significant increases in performance and a more frequent use of heuristics (Komorek et al., 2006).

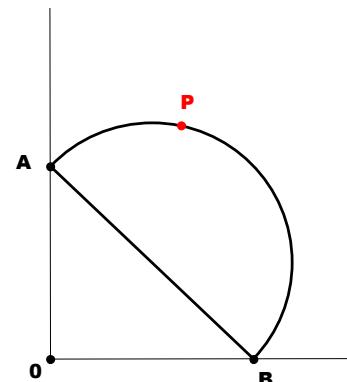
Results of the work products

The work products submitted by the teachers at the end of the school year were evaluated by means of a category system, the results were reported to the teachers. 17 teachers submitted 38 work products in total (8 working sheets, 5 particular problems, 4 learning controls, 8 long-term homeworks, 13 lesson designs). The following aspects were taken into account for the evaluation of the work products: *subject-related criteria, aspects of goal orientation* (transparency of the goal for the students, clearness of the goal by the teacher), *motivation potential, internal differentiation, cognitive activation* (especially variation of degree of difficulty and task type), *student activities* and *self-regulation*. The results of the submitted work products prove the successful concept implementation by the teachers (Komorek et al. 2006). An exemplary work product (shortened version) of a teacher from the problem-solving group (PS) shows a specific implementation of the concept (in the sense of field orientation). The task, following an upgrading of requirements, starts with low requirements and becomes more complex with every subtask. Source of the task is a so-called “two-minute task”, taken from a Hungarian TV-show in the sixties.

Original task from a TV-show:

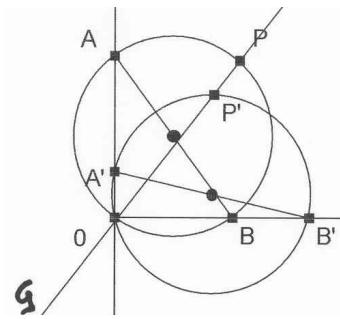
“The semicircular disc glides along two legs of a right angle. Which line describes point P on the perimeter of the half circle?” (Engel, 1998). The problem for students modified by the teacher is:

- Make a construction of beer coasters or similar materials to visualize the problem.
- Which line describes point P?
- Explain the form of the curve.



A student who was working with a dynamic geometry software explained her solution of the two latter subtasks as follows:

Eva: I didn't produce a curve but a line. All points of P are on this line. When we move the segment AB and A'B' on the coordinates, line G is always following the movement in a way that it goes through point P and P'. I do not have an explanation for line G, but I believe that whenever a point is attached to G and AB or A'B' is moved, it describes a curve.



The teacher reflects upon the use of the task as follows:

Teacher: Experimental homework is useful and arouses the curiosity of many students. The goal was reached to make a maximum of students deal with this most demanding geometric problem. The written documentation of the problem-solving process revealed a multitude of individual perceptions, approaches and procedures of the students and also showed the causes of possible problems.

Results of the lesson reports

1296 lesson reports were presented by 38 teachers who had documented their lessons in 15 to 41 periods. From the items collected in a four-stage answer format (3: "very true" to 0: "not true") the following five scales were established:

- Inner and outer mathematical problem-solving (Cronbachs $\alpha=.89$; 6 Items)
- Use of strategies – reflection on action (Cronbachs $\alpha=.86$; 4 Items)
- Internal differentiation – individual learning (Cronbachs $\alpha=.80$; 4 Items)
- Formation of exercise processes (Cronbachs $\alpha=.73$; 6 Items)
- Accomplishment of lesson target (Cronbachs $\alpha=.86$; 4 Items)

The first three scales were subsumed in a superordinate scale "problem-solving and self-regulation" (Cronbachs $\alpha=.94$; 14 Items). This scale includes elements for the enhancement of partial actions of problem-solving, the integration of heuristic and self-regulative elements in the lessons and the heterogeneity in class. The following is focussed on this scale.

Analyses of trends in connection with the further training method (CB, WB, NO) show that curricular coaching or webbased coaching are considered to be a cause for significant positive trends towards the implementation of the concept. On the other hand the input in the form of a single compact course does not create a positive trend, a certain starting level in the context of the enhancement of basic competencies notwithstanding. Figure 1 illustrates the trends of the curricular further training group (CB) and of the further training group which was not supervised during the school year (NO).

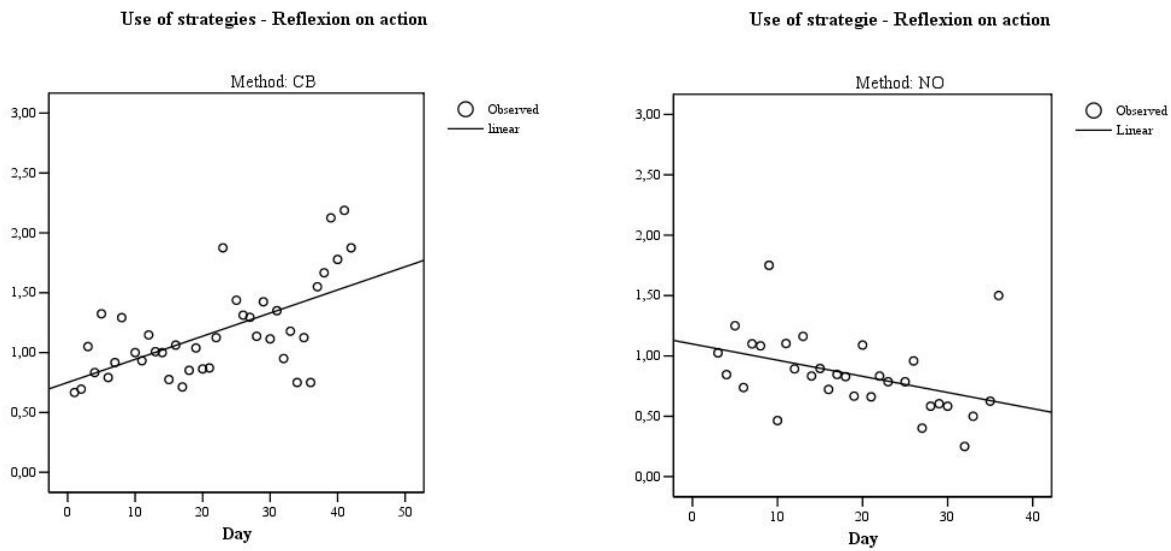


Fig 1: Trend analysis (scale: “use of strategies – reflection on action“)

As a processual instrument the lesson report reveals the effect of a concrete intervention, resulting of an additional further training course. Coaching throughout the school year should influence the variables of the lesson report in a way that the participation in this further training produces a significant difference between self-perceived teacher action after the further training course at the beginning of the school year (first intervention phase) and after the intervention (second intervention phase). It was not possible to run a classical baseline phase without placing excessive burden on the teachers. The effect of the additional training on the curricular-based group (CB) was analysed with an *intervention analysis* (ARIMA (1st order autoregressive model)). On the 18th day of the study a significant training effect can be observed (figure 2).

As for the integration of heuristic elements it is stated that in the Gymnasiums heuristic tools as well as strategies and principles are adopted. The teachers of other school types are rather focussing on heuristic tools. Most frequently used heuristic tools were the informative figure (122), table (86), graph/diagram (64) and strategies like forward and backward working (121). The frequency of mentioned heuristics by the teacher is put in brackets.

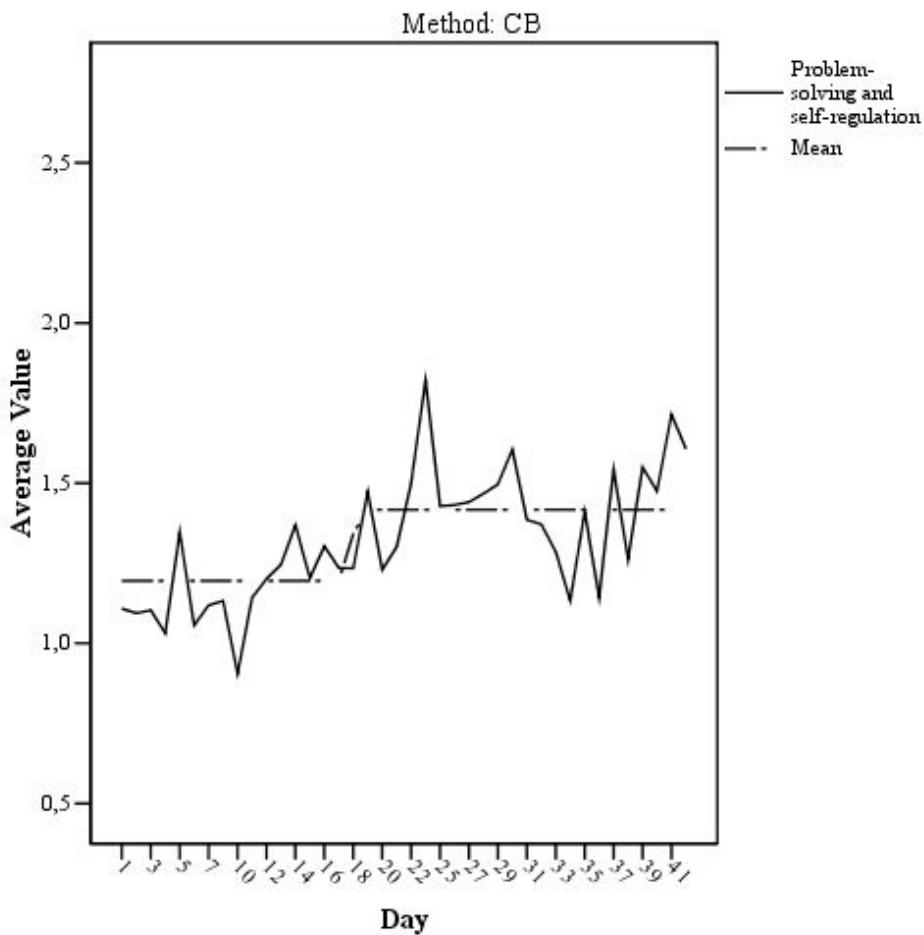


Fig. 2: Intervention analysis (scale: “problem-solving and self-regulation“)

Conclusion and outlook

The results of all instruments prove the successful implementation of the concept. The work products of teachers are suitable to show competencies acquired in the further training. Due to the positive results of this study the collection of work products and their evaluation became part of the e-learning training courses (www.prolehre.de), organized by the Technical University Darmstadt since 2005. The lesson report as processual instrument allows to draw conclusions from the self-perceived action competence with respect to the implementation of the concept. The results show that in addition to a single compact course further coaching during the school year is necessary to enhance the integration of the concept ideas. The multi-perspectivity of the applied tools allows a connection of teacher knowledge and practical action with regard to the learning of problem-solving and self-regulation according to the proposal of da Ponte et al. (2006).

The results of a follow-up study (one year after the further training) with 11 project classes are currently evaluated. They are expected to reveal the long-term effects of the further training courses on the students.

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MATHEMATICS EDUCATION AND TORRES STRAIT ISLANDER BLOCKLAYING STUDENTS: THE POWER OF VOCATIONAL CONTEXT AND STRUCTURAL UNDERSTANDING¹

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Torres Strait Islanders (TSIs) are Australian Indigenous people whose lands comprise the Islands between Australia and Papua New Guinea. As a consequence of past government policies, they share with Aborigines social disadvantages (Fitzgerald, 2001) that lead to mathematics underperformance and limit education, employment and life chances. This paper reports on an action-research collaboration between our research team and a vocational teacher to enable mathematically-underachieving TSI adult students to meet the requirements of a blocklaying course. It discusses Indigenous vocational mathematics, describes the theories behind our VET Project and the findings of the blocklaying collaboration, and reinforces the importance of vocational contexts and structural understanding (Sfard, 1991) when teaching vocational mathematics.

Demand for skilled trades' workers, such as blocklayers, has increased in response to Australia's economic growth in the past decade. The training of these workers is managed through a national system of post-compulsory education (post Year 10) called Vocational Education and Training (VET). Standardised national training packages, developed through consultation with industry bodies, describe the skills, knowledge and assessment needed to perform a trade. Most of this training is delivered by state government Technical and Further Education (TAFE) institutes, although schools and private institutes are also involved. The training packages do not stipulate learning strategies nor take account of cultural difference. It is up to the individual TAFE teachers to determine the needs of their students and the "how" of their teaching. This means that each TAFE Institute has different methods of catering for low achieving and Indigenous students, particularly for students whose mathematics is inadequate for their trade.

Indigenous students and VET. Indigenous students have the lowest retention rates in the Australian school system, often leaving school before completing Year 10 with lower levels of mathematics than non-Indigenous students (Bortoli & Creswell, 2004; Queensland Studies Authority (QSA), 2004). This has been attributed to racism, remoteness, English as a second or third language (ESL), social factors (Fitzgerald, 2001) and systemic issues including non-culturally inclusive forms of teaching, curriculum and assessment (Matthews et al., 2005). As cultural knowledge, mathematics is taught from a Eurocentric position which means that Indigenous and other minority cultures experience alienation from and conflict with it (Matthews, et al., 2005). This flows over to post-compulsory education, where Indigenous students experience anxiety in regards to mathematics due to prior negative schooling experiences (Katitjin,

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McLoughlin, Hayward, 2000). Thus, Indigenous retention in post-compulsory education is low and few complete senior secondary subjects at a level that enables them to enter University (QSA, 2004). Indigenous students enter VET earlier (the proportion of Indigenous students aged 16 years or less entering VET is higher than non-Indigenous students) and are less educated (over half at or below Year 10 on entry to VET) and more likely to be ESL and remote (DEST, 2003).

The VET Research Project. Our interest in mathematics education and VET has arisen for two reasons. First, most VET training packages contain mathematics requirements and assessment that cause problems for TAFE students with low mathematics knowledge. Gaining mathematics skills has been identified as a central equity issue in education, obviously because mathematics proficiency is “required not only in educational contexts but also in employment” (Louden et al., 2000, p. 202). Second, VET training requires students to integrate prior and new mathematical understandings in a workplace context very different to schooling (Martin, LaCroix & Fownes, 2005). This offers an opportunity for new approaches to teaching mathematics that may overcome the legacy of long-term school failure. This interest in VET mathematics education has led to a VET Research Project¹ in which we collaborate with TAFE teachers across Queensland in a variety of courses and programs to study effective ways to teach mathematics to underperforming TAFE students. This paper reports on one of these collaborations, a study of TSI TAFE students learning blocklaying mathematics.

THEORETICAL FRAMEWORKS

The VET Research Project’s aim is to study the value of utilising vocational contexts to teach mathematics to structural understanding (Sfard, 1991). The methodology is qualitative, interpretative and intervening, using case-study approaches to investigate collaborative action research between us and TAFE teachers in improving mathematics education. The project’s participants consisted of the teachers and their students. It has two theoretical frameworks, namely, vocational context and structural mathematics.

Vocational context. The first imperative of the Project, and the TSI collaborative study was that mathematics instruction should be situated within a vocational context. This came from the findings of our first VET study (Baturo & Cooper, 2006) which focused on Aboriginal horticulture students at a community secondary school in a rural area. Initially, these students studied horticulture in the morning with a TAFE teacher and returned after lunch for mathematics with a school teacher. Then the two components were integrated and the students studied horticulture and the associated mathematics all day with the TAFE teacher. Our findings were that: (a) separating mathematics and horticulture was unsuccessful (most students simply did not attend the mathematics lessons); (b) teaching mathematics within a horticultural context was successful as long as the TAFE teacher was given content and curriculum support; and (c) horticultural and mathematics study was most effective if it integrated authentic tasks such as constructing a fence for a park (“doing real work that really matters” – Keller, 2007, p. 2) with virtual simulations of the same work. For example, the horticulture students found difficulty in determining the number and placement of fence posts even with a calculator because

they did not understand the situation as division. A computer simulation (using PowerPoint) in which fences could be built with virtual posts and discussed in terms of fence distance, post separation and number of posts was found effective in following up real life fencing situations.

It was evident to us that the Aboriginal students preferred to meet mathematics within an authentic vocational rather than school-like context (as described in Martin et al., 2005). This is in line with growing global recognition of context-based initiatives such as authentic pedagogy (Newman & Wehlage, 1995) and work-integrated learning (Gibson et al., 2006) and the continuing power of seeing cognition in situation (Brown, Collins, & Duguid, 1989; Lave, 1988). These approaches (gathered together under the title “vocational context” in this paper) posit that knowing and thinking should be considered as an interaction between an individual and physical and social situations that allow for meaningful constructions and statements about learning (Wenger, 1998) and develop schemas that mediate between experience and learning (Derry, 1996). They involve a curriculum integrated with VET needs, work components to allow learning through experience, relevant placement in VET activities, and well-defined logistics for organising, coordinating and assessing students (as Bates, 2005, proposed for teacher education). Community involvement and ownership that is a consequence of vocational contexts has been identified as the single most important factor of Indigenous success in VET courses (O'Callaghan, 2005).

Structural mathematics. The second imperative adopted by the Project was to always take mathematics instruction beyond procedural to structural understanding, at the same time contextualising the instruction by incorporating Indigenous culture and perspectives into pedagogical approaches (Matthews, Watego, Cooper & Baturo, 2005). The focus on structure was due to our prior success with this approach when educating adult Indigenous and non-Indigenous teachers and teacher assistants (Baturo, 2004; Baturo & Cooper, 2004). It was also reinforced in our first contact with the blocklaying teacher where he stated that his students had problems converting from mm to m. He had introduced this conversion by modeling 1 m as the length from 0 mm to 1000 mm on a measuring tape and then explaining that a decimal point preceded the amount when converting mm to m. This had not worked because many students then believed that 6 mm was 0.6 m, 60 mm was 0.60 m, and 600 mm was 0.600 m.

THE BLOCKLAYING COLLABORATION

The government centre in the Torres Strait has a small TAFE campus staffed by non-Indigenous and Indigenous people, including elders. This campus runs programs on skills needed in the islands: general construction, marine studies, horticulture, business, art, cultural studies and general adult education. The general construction program takes young, predominantly-unemployed, TSI men and trains them for 1.5 years in blocklaying. On completion, students go to Cairns (1000 km south on the mainland) and can be apprenticed, become labourers for builders, or use their skills to build for themselves and their Island Communities. Our collaboration with the TAFE teacher of the blocklaying course involved finding ways to develop TSI students' understanding of

the mathematics concepts and processes embedded in blocklaying situations sufficient for the students to be successful in building block structures and in meeting certification.

Teacher, course structure and students. The TAFE blocklaying teacher was not Indigenous but was a highly qualified master builder with builder-training certification. We were impressed that he was already teaching using vocational contexts: (a) building walls and small buildings with blocks and soft mortar in an on-campus construction centre in a way that allows constructions to be dismantled and blocks reused; and (b) taking the students to different Islands to build actual structures (e.g., boat storage sheds, retaining walls, a small bridge, school entrance gates) for the community and private businesses. He did all the teaching for the blocklaying course and integrated mathematics into his blocklaying teaching. As well, he was organising partnerships with local industry and island companies and councils to gain full-scale professional projects with which the students could network future employment.

He emphasized learning to build personal and community capacity as much as to gain certification: *I say to them, do the course and you will be able to build your own house even if you don't go on and get certification.* He had built up a strong personal relationship with his TSI students that went beyond classroom contact hours. He had no training in mathematics education; not surprisingly, he saw mathematics teaching in procedural terms (as is evidenced by the measuring tape example described earlier).

The official course through which blocklaying was taught was Certificate 1 in General Construction (as there were not enough master builders in the Torres Strait to enable apprenticeships) and ran for 18 months. The course was designed so that, at its end, students would have sufficient practical experience to enter into an apprenticeship and get recognition that would be equivalent to two years as an apprentice.

The blocklaying students varied in age from 18-26 years and all were Indigenous. Some students came from the outer islands and were selected by their Island's councils and elders to become builders for their communities. Of the ten students interviewed, five had completed Year 12; three had completed Year 11; and two others had finished Years 9 and 10 respectively. However, the mathematical ability of most students was not much more than mid elementary school (as is indicated in the mathematics problems they were having). This caused problems when mathematics was encountered in the course and led to early withdrawal and failure. The mathematics in the blocklaying course was supported by module workbooks but the teacher recognized quickly that these workbooks were not conducive to Indigenous student learning styles or needs.

Intervention. We undertook the collaboration at two levels, professional learning and planning sessions with the TAFE teacher, and model lessons with the students. Both the sessions and the lessons focused on teaching structural understanding of the students' mathematics-learning difficulties identified by the teacher as can be seen in the following summaries of interventions for three of these difficulties.

- 1) *Whole and decimal numbers and conversion from mm to m.* A kinaesthetic activity from Bاتuro (2006) reinforcing multiplicative structure in which 9 students in a row

are labeled as place value positions in periods of three (1-10-100 ones, 1-10-100 thousands, and 1-10-100 millions). Other students labeled as digits (say 7) move between them as the remaining students use calculators to determine the conversion rates (e.g. moving from 10 ones to 1 thousand is multiplying by 100). This is repeated with the periods relabeled as mm, m and km and reinforced by metric slide rules where a cardboard strip labeled mm, m and km is slid along a place value chart allowing, in turn, mm and m to be in the ones position.

- 2) *Proportion and measurement to interpret house plans drawn with a scale.* Two PowerPoint virtual activities (Baturo & Cooper, 2006) were designed. The first reinforced the relationship between plan and reality in terms of scale. Students changed a copy of a rectangle in relation to the original as directed by a scale and then translated this scale to a change between mm on plan and m in reality. The second reinforced building walls with blocks. Students used virtual pictures of whole, half and three-quarter blocks, doors and windows to construct walls and determine numbers and types of blocks in relation to area.
- 3) *Area and volume of footings.* A PowerPoint activity was designed for whole class discussion to enable area formulae to be discovered and translated to volume formulae. Students manipulated pictures of tiles to relate length and width to area and then added layers of tiles to a starting base to relate height with area of base.

Student, teacher and administrator responses. The professional learning sessions and the model lessons were videotaped and the blocklaying teacher and TAFE administrators were interviewed. The interviews indicated that the blocklaying course was considered a success by the students, the teacher and administrators. To date attendance and achievement was higher than expected in the course and the students were able at the end of the program to build with blocks. As a consequence, the course was being maintained at a time when other TAFE institutes were curtailing such activities. The TAFE teacher was strong in his praise for the students, saying most were employable because they could build with blocks sufficiently well to *make money for an employer*.

With regard to vocational contexts, responses from students and staff indicated strong support for this teaching approach, particularly when it was done through authentic tasks that supported the community. Many students stated that they preferred learning on the job site to in the classroom. They felt they could understand mathematics if it was being implemented at the job site; as Student 1 stated, *I like it on-site because you can see it and you know what it is*. Of course, they were pragmatic; as Student 2 said when asked about classroom or site, *On the site ... but a bit of both hey*.

The students were particularly motivated by gaining skills that their Islands could use. Providing for their community and helping community members who had helped them in the past was important to these students. When asked if he would go to Cairns, Student 4 stated, *I'd probably stay here and work out on the outer islands*. When asked his reason, he stated, *Help the people and help me*. When asked what he wanted to do and where, Student 2 said, *I want to become a contractor, I want to*

have a chance to give back to people who have helped me, while Student 5 stated, I just want to pass this course ... and start my own business ... Yeah on TI, there's a couple of outer island people that I want to help get into TAFE and do block work.

It was also evident that a strong relationship existed between teacher and students and was a key factor in the success of the course. A number of students remarked how they appreciated the encouragement and support provided by their teacher; Student 3 stated, *He's a pretty good bloke. He's not too tough, but when he wants us to do something ... He's a good teacher though*, while another said, *Yeah, he's alright. He doesn't discourage us if we do something wrong and there's always encouragement from him*. The importance of building a relationship was supported by the teacher who said, *Once I've built relationships with them, which takes about two weeks, they start to relax a bit with me ... When [one student] first showed up I actually threw him out of the yard because he wouldn't speak to me. He wouldn't communicate. He was very withdrawn ... Once he decided that he'd get to know me I had no problem. That's the same with all the boys. I have to build a relationship with them before I can get them to do anything.*

With regard to structural understanding of mathematics, the students and the teacher appeared to find the modeling of the structural ideas useful for their blocklaying mathematics. Analysis of the video tapes showed that when students participated in the kinaesthetic activity to learn about whole and decimal numbers, the level of engagement in the classroom increased. Mathematics related discussion between students, teachers, and our research team was also greater than at other times during the modeling. The video tape also showed students highly engaged when they were presented with the PowerPoint activity on volume. Evidence of this was seen as one student controlled the PowerPoint slides and the rest of the class aided him with verbal encouragement and suggestions. When we asked at the end of the modeling session, *Do you like the computer stuff?* Most students called out yes, and one particular student exclaimed, *For sure!* One student remarked to his teacher later in the day that for the first time, he understood volume; the teacher described it, *I was just talking to a couple of boys, and they said that they understand the volume better on [the data projector] ... I did exactly the same thing except I did it on the board. They've understood this better. [One student] who I thought had grasped it very well, just said to me, I understand volume now.*

The feedback from the blocklaying teacher was even stronger; he changed his opinion of how to teach mathematics as a result of the intervention. At the end of the first research visit, he approached a member of our team and admitted that he initially had concerns that the visit would be a waste of time but he was wrong. He said that the professional learning and planning sessions and modeling lessons had changed his opinion and that the approaches on offer were valuable to him and would prove valuable to his teaching. He stated that he was looking forward to the next intervention and would implement the ideas and approach with his students.

DISCUSSION AND IMPLICATIONS

The central implication of the study is that vocational contexts and structural learning of mathematics is effective in VET situations. The collaboration with the blocklayers showed the efficacy of vocational context in line with the arguments and findings of Baturo & Cooper, 2006; Brown et al., 1989; Gibson et al., 2006; & Newman & Wehlage, 1995. Both teacher and students strongly supported the vocational context approach for learning mathematics. The success of the blocklaying course in comparison to other TAFE Indigenous initiatives is a strong validation of the effectiveness of the approach. One of the reasons for success lies in the course's organization. Discussions with the blocklayer teacher revealed that he met the four areas that Bates (2005) argues are important for a successful work integrated program: VET integrated curriculum; experiential learning; VET placements and well-organised and implemented course. The collaboration also showed the importance of the contexts being authentic in relation to community involvement (O'Callaghan, 2005) and being work that matters (Keller, 2007). The students positively responded to learning that provided them with skills to improve their own lives and their community. Also, the students responded well to the positive relationship with their TAFE teacher, considered an important attribute of successful Indigenous programs (Baturo & Cooper, 2006). They had developed pride in their identities as blocklayers, also a basis for success in Indigenous learning (Matthews et al., 2005).

The collaboration also validated the importance of mathematics in VET (Louden et al., 2000) and the value and significance of structural learning of mathematics (Sfard, 1991; Baturo & Cooper, 2006) to students' successful use of mathematics in workplace situations. Although the blocklaying students experienced anxiety towards mathematics (Katitjin et al., 2000), they were able to understand the mathematics in blocklaying contexts, most likely (as their interview responses disclosed), because it was different to schooling (Martin et al., 2005). The collaboration also reinforced the efficacy of virtual (PowerPoint) mathematics materials (Baturo & Cooper, 2006) as shown in the students' reactions to the modeling lesson.

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INTERDISCIPLINARY LEARNING AND PERCEPTIONS OF INTERCONNECTEDNESS OF MATHEMATICS

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This paper studies the effect of interdisciplinary project work on Singapore students' perceptions of mathematics. Interdisciplinary project work aims to prepare students for the knowledge-based economy, emphasise links within and between school subjects and core skills such as communication. Two scales measuring perceptions of the interconnectedness of mathematics were completed by 409 students aged from 12 – 14, in 3 schools, before and after participating in a 12 – 16 week project. Amongst statistically significant changes was a relatively moderate increase in scores on the interconnectedness scale after project work. Students in different ability streams perceived and used interconnectedness in different ways both before and after the project work. Teaching emphasis on conscious integration of subject areas is needed.

INTERDISCIPLINARY PROJECT WORK IN SINGAPORE

Interdisciplinary project work (PW) was introduced as an educational initiative in Singapore in 1999 to prepare students to meet the demands of a knowledge-based economy (CPDD, 2001). To stay relevant in such an economy, in-depth knowledge of specific subjects is insufficient. Students need to integrate ideas from various disciplines for problem solving. Curriculum planners in various parts of the world (e.g., McGuinness, 1999; NCTM, 1995), including Singapore, have begun to emphasise explicit links between school subjects. PW is seen as a platform for incorporating core skills and values, and integrating subject-specific knowledge in innovative ways (Chan, 2001).

PW contributes to the Singapore vision of *Thinking School Learning Nation* (Quek, Divharan, Liu, Peer, Williams, & Wong et al., 2006). Coping with a knowledge-based economy requires meaningful integration of various subject-specific content areas for problem solving. Thus, the Singapore Ministry of Education is moving away from sole dependence on paper and pencil examinations, and including holistic, student-centred learning activities and authentic assessment modes. PW is considered as a “paradigm shift” (Quek et al., 2006, p. 14) from teacher-dependent to student-initiated learning. The learning outcomes of PW are stated in 4 domains: communication, collaboration, independent learning, and knowledge application, which include emphasis of the interconnectedness of what is learned (CPDD, 2001).

PW tasks are mainly adapted from Ministry resources or designed specifically by teachers according to yearly themes set by schools. Every PW task anchors in at least two or more curriculum subject areas. Ideally a team of at least two teachers from these subject areas are allocated to the class. A typical PW task consists of a major driving question involving a real-life problem or scenario. Critical intellectual activities are generated from the task, with purposeful integration of content from its various anchor

subjects. PW is different from the interdisciplinary or integrated curricula espoused by many schools in the West (see Jacobs, 1991), where content learning derives from thematic-related project tasks and investigations. In Singapore, the necessary content and skills are taught during traditional subject-specific lessons, and PW teachers provide “just-in-time instructions” (CPDD, 2001) so that groups can work independently and creatively while applying the knowledge and skills taught.

Aims of this study

The work reported here is part of a mixed methods study into the nature of the thinking that students engage in during PW and the effects of PW on their mathematical learning. Close analysis of selected groups of students as they work on a project enables a study of the interplay between cognitive, metacognitive, and social processes during PW.

A quantitative survey-based study has also been conducted to measure changes in various affective measures as students participate in PW, and to observe differences between groups of students. A review of the literature relating to links between affect and problem solving, of goals and expectations of PW in Singapore, and extensive statistical analysis of draft scales led to the identification of three constructs as being especially relevant to interdisciplinary learning:

- *confidence in mathematics scale*
- *value of mathematics scale*
- *interconnectedness of mathematics (ICS) scale*, incorporating 2 subscales of inter-subject learning (ISL) and beliefs and efforts in making connections (BEC).

This paper will report only results from ICS scale. Results for the other scales can be found in Ng and Stillman (2006). Perceptions of the interconnectedness of mathematics have not been measured before, to our knowledge. We will report on the construction of the scales, differences between groups (gender and educational stream), and the impact of PW on students’ perceptions of ICS.

INTERCONNECTEDNESS IN MATHEMATICS

One aim of implementing PW in Singapore schools is to make explicit the interconnections of subject-specific knowledge so that learners can “transcend subject boundaries and make connections between the various subject areas” (CPDD, 2001, p. iii). Proponents of the interdisciplinary curriculum (e.g., NCTM, 1995) emphasise the importance of drawing links between content and skills of school subjects to encourage holistic learning.

Beliefs, self-confidence, and value in mathematics have been examined by researchers (e.g., Schoenfeld, 1989) to draw inferences on students’ affect and problem solving. Though there are other Singapore studies on PW, Tan (2002) was the only study to date relating mathematics attitudes to PW. He found that a problem-based approach to PW improved most of the six dimensions of mathematical attitudes studied from his

all-male sample in two educational streams. However, he did not look into students' perceptions of the interconnectedness of mathematics.

Despite growing emphasis on interdisciplinary learning in curricula, no literature to date has been dedicated to quantifying the perceptions of interconnectedness between subjects. This study began by examining the Singapore PW goals and identifying three components of interconnectedness namely, how students perceive (a) mathematics content and skills in relation to other subjects; (b) the usefulness of mathematics in understanding other subjects; and (c) the complementary relationship between mathematics and other subjects in problem solving. These components represent a continuum, from awareness of interconnectedness, to consideration of use, to actual use.

Items were constructed for these three components. For example, for the first component, item BEC2 (see Table 1, which gives the final version only) may indicate high personal sensitivity to interconnectedness of mathematics. The second component arises from assertions among interdisciplinary proponents that students' understanding of one subject can be reinforced by another (Jacobs, 1991). Items focussed on links such as whether students recognise the possibility of transferring knowledge across subjects (e.g., ISL4). Knowing that interconnectedness exists between subjects does not imply action, however. Boix Mansilla, Miller, and Gardner (2000) assert that interconnectedness require that students combine discipline-based knowledge to solve problems, so the third component (e.g., ISL6) measures the application of mathematics and other subjects to solve real life problems, which are usually interdisciplinary in nature. High scores here imply flexible integration of knowledge across disciplines, the epitome of the goals of interdisciplinary education.

Although conceived as three components, extensive trialling of the scales, as described below, resulted in two subscales for ICS, with good statistical properties. We tried to encapsulate the differences in the factors in the two names *Inter-subject Learning (ISL)* and *Beliefs and Efforts in Making Connections (BEC)*. All items from BEC scale derived from those proposed for the first component above. ISL items derived from all 3 components. The final items and scales are shown in Table 1.

METHODOLOGY

Construction of the scales

The first version of the confidence, values and interconnectedness scales comprised of 45 five-point Likert items. Two expert panels from Singapore and Australia vetted the initial items. The three scales were constructed in three phases with 283 students in the target age range (12 – 14) from 7 Singapore secondary schools. Care was taken to trial with students of a range of English language abilities from government neighbourhood schools, to be similar to the 3 schools which undertook the PW study.

First, items were face-validated by 9 students from three streams in individual interviews, who explained their responses and rephrased problematic items. Rephrased

versions were re-tested on subsequent interviewees. The 45 items were then administered to another group of 36 students twice within one week and 13 students with high response inconsistency attended individual face-to-face interviews to identify confusing statements for deletion. The scales were reduced to 41 items. Second, two schools held a large-scale trial ($n = 204$). Factor analyses were conducted separately on the three scales. Varimax rotation revealed a total of eight subscales, including two (ISL and BEC) for ICS as noted above. Table 1 shows basic statistics for the scales. Third, test-retest reliability was checked with 34 students who responded twice in one month. All subscales displayed moderate to high stability. Future studies could add to the development of the ICS scale pioneered by this study. Full details of the development of the scales are given by Ng and Stillman (2006). In the final questionnaire, the items were given with others that elicited background information. Students were told to respond with reference to their most recent classroom experiences.

Interconnectedness of Mathematics Scale (ICS) (Variance explained = 44.85%)

Inter-subject Learning (ISL) ($\alpha = 0.787$; Test-retest $r = 0.62$)

- ISL1 I have used math while working in another subject before.
- ISL2 I can see links between some math topics and other subjects.
- ISL3 Sometimes I use math to help me understand another subject.
- ISL4 I can use math to help me learn another subject better.
- ISL5 I use another subject to help me learn math sometimes
- ISL6 Sometimes, I combine what I know from math and other subjects to solve problems.
- ISL7 Math may share some common topics and skills with other subjects.

Beliefs & Efforts in making Connections (BEC) ($\alpha = 0.587$; Test-retest $r = 0.53$)

- BEC1 I don't try to make connections between math and other subjects when I learn.
- BEC2 It is important to relate math to other subjects when learning.
- BEC3 I find learning more meaningful when math and other subjects have common topics.
- BEC4 Math has no connections with the other subjects I am studying.

Table 1. The items of the Interconnectedness of Mathematics Scale

The project work

The main study centred on the implementation of PW in three Singapore schools. PW lessons are incorporated into the normal timetable and students work in groups of 4 or 5, consulting teachers at least once a week for about 13 weeks, and they also work outside class time. Sixteen classes undertook the PW with a total of 632 students, of

whom 409 agreed to submit data to the study. In Singapore, students are streamed on entry to secondary school. The 16 classes were from the highest (Express) and middle (NA = normal-academic) overall abilities in language, mathematics, and science. One school had one teacher (mathematics) for PW in each class and the others had two.

The first author created a PW task on environmental conservation. Ng (2006) describes the task “Designing an environmentally friendly building” in detail. The task formally combined mathematics, science and geography and aimed to enhance students’ environmental consciousness. The flexibility, scope, and breadth afforded by the task enabled students from differing ability streams to participate. Extensive design and support materials were provided for the teachers. For their final products, students constructed physical scale models of their buildings using recycled materials. Mathematical concepts and skills include estimating dimensions, making appropriate scale drawings, constructing 3-D scale models, costing and evaluating.

The classes used the project-based approach. They were given a scenario to explore, brainstorm, and plan their own schedules for completion. Students collaborated in decision making, research, and constructing the prototypes of their buildings. Assessment of the task evaluated the process (e.g., drafts) and products produced. Each member of the group presented a portion of their work orally to the class. Individual and group scores were awarded based on rubrics.

RESULTS

Table 2 gives the means and standard deviations for the scores on the two subscales of interconnectedness (ISL and BEC) for the whole sample and various subsamples. All the means are in the range 3.25 – 3.75, indicating that these groups of students were on average about halfway between neutral and agreement on the five point scale. The overall ISL mean and the ISL means of all of the subgroups in Table 2 improved after the PW experience. The overall BEC mean shows very little change after the PW experience, with some the subgroups making a small increase and some making a small decrease from pre-task to post-task.

The statistical significance of the differences in the table were tested with a general linear model (GLM) i.e. a multi-factorial repeated measures ANOVA. The repeated measures were the dependent variables (ISL and BEC) scores pre-task and post-task (time factor), and there were 3 independent factors: gender, stream and school.

School differences. The school differences were not statistically significant, and so students from the three schools are put together to make one sample.

Pre-task to post-task differences. The change in ISL on the whole sample was statistically significant at the 5% level (Wilks’ Lambda = 0.966, $F(1, 391) = 13.760$, $p = .000$, $\eta_p^2 = .034$). The effect size of 0.034 (measured by η_p^2) is between Cohen’s (1988) limits of 0.01 for small and 0.06 for moderate effect. The change in BEC from pre-task to post-task was not statistically significant (Wilks’ Lambda = 1.00, $F(1, 390) = 0.014$, $p = .906$, $\eta_p^2 = .000$).

Gender differences. The BEC subscale showed a small but significant gender difference favouring males ($F(1,1) = 4.460$, $p = 0.035$, $df = 1$, $\eta_p^2 = .011$) but the ISL scale did not ($F(1,1) = 0.032$, $p = 0.858$, $df = 1$, $\eta_p^2 = .000$). Time-gender interaction effects were not significant, implying that gender was not a factor for reaction to the PW experiences, but a factor for between-subject differences.

Stream differences. There were significant differences of relatively moderate effect size between Express and NA students for both ISL ($F(1,1) = 10.891$, $p = 0.001$, $df = 1$, $\eta_p^2 = .027$) and BEC ($F(1,1) = 11.713$, $p = 0.001$, $df = 1$, $\eta_p^2 = .029$), with express students scoring higher. There was no time-stream interaction effect for either ISL or BEC, indicating that PW affected the attitudes of each stream in the same way.

Sample Groups	Inter-subject Learning (ISL)				Beliefs and Efforts at making Connections (BEC)			
	Pre-task		Post-task		Pre-task		Post-task	
	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
All (n = 409)	3.52	0.58	3.64	0.53	3.48	0.58	3.49	0.60
Males (n = 206)	3.51	0.58	3.66	0.60	3.52	0.61	3.49	0.65
Females (n = 203)	3.53	0.58	3.62	0.47	3.44	0.56	3.48	0.54
Express (n = 295)	3.59	0.57	3.70	0.50	3.54	0.56	3.57	0.56
NA (n = 114)	3.35	0.57	3.50	0.59	3.31	0.61	3.27	0.64
School 1 (n = 151)	3.56	0.51	3.61	0.49	3.46	0.59	3.44	0.60
School 2 (n = 94)	3.40	0.59	3.54	0.64	3.38	0.59	3.36	0.66
School 3 (n = 163)	3.56	0.63	3.73	0.50	3.56	0.57	3.60	0.53

Table 2. Means and Standard deviations for ISL and BEC scores.

DISCUSSION AND CONCLUSION

This study has pioneered the quantitative measurement of perceptions of interconnectedness of mathematics, providing information about students in various groups, and before and after an experience of project work which has appreciation of interconnectedness as one of its goals.

Through a careful process of test development, it was found that perceptions of interconnectedness could be summarised in two factors which had good statistical properties: *inter-subject learning* (ISL) and *beliefs and effort in making connections* (BEC). Further research by others on the measurement of perceptions of interconnectedness would be welcome, to test this factor analysis beyond Singapore secondary school students. Since increasing appreciation of interconnectedness and promoting interdisciplinary learning are common goals of school systems, a robust analysis of the concept should be very useful.

There were differences of relatively moderate effect size between the two ability streams, favouring higher ability students, before and after PW. Express students tend to perceive the interconnectedness of mathematics more than NA students, and are likely to make efforts at making connections such as engaging in using mathematics for inter-subject learning. Students in different ability streams seem to define and make use of the connections between subjects for their learning in different ways. Although the ISL and BEC subscales are still in developmental infancy, information on this would assist future facilitation of interdisciplinary learning for students in different educational streams.

Administration of the two subscales showed that there was a small gender difference favouring males on BEC, but not on ISL. This provided further evidence that the two subscales are indeed different, and added further information to the knowledge of gender differences in mathematics in an Asian setting.

In accordance with the goals of project work, there was a small improvement of scores to ISL after participating in the interdisciplinary project. Students after the project work were somewhat more likely to appreciate mutual reinforcement of learning among mathematics and other subjects. They did not, however, report an increased effort to make such connections (BEC). We have no good explanation of why there should be an improvement in one of the subscales but not the other, although the qualitative data gathered in the study may soon provide some clues.

An improvement in appreciation of interconnectedness would be expected because of the nature of the project experience, which required students to use mathematics in conjunction with other subjects. In two schools, students had teachers from different subject areas facilitating PW groups. There are also some reasons why the effect may have been small. About half of the students had done project work before, and about a third had used mathematics (usually only arithmetic) in their previous PW, so they may already have developed some ideas about interconnectedness.

Classroom observations and teachers' reports were that the open task was appropriate for classes of both ability levels, because the students were largely able to set the mathematical demands of the task for themselves. The main mathematical difficulties were in fixing the scales for the building and making the 3-D model from 2-D drawings. Often these mathematical tasks were delegated to 'experts' within the group, so that not all group members necessarily had experience of using mathematics in the real, interconnected situation. This may have contributed to the small effect size on interconnectedness scales. Teachers have to be aware that relevant and conscious knowledge application of subject content and skills into the interdisciplinary task is not automatic. Similarly, successful integration of the various knowledge and skills from different subjects, used together in the task depends heavily on the nature of teacher facilitation and close monitoring of student progress. The challenge is to strike the right balance in providing guidance within the scope and depth of the theme.

Finally, we acknowledge that there are many goals of PW beyond those on which this paper has reported. It is not possible to judge the success of either this task in particular, or PW in general, from the interconnectedness measures alone.

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THE TREATMENT OF ADDITION AND SUBTRACTION OF FRACTIONS IN CYPRIOT, IRISH, AND TAIWANESE TEXTBOOKS

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In this paper, we report on an analysis of the treatment of addition and subtraction of fractions in elementary mathematics textbooks used in Cyprus, Ireland, and Taiwan. For this purpose we developed and applied a framework to investigate the learning opportunities afforded by the textbooks. We found key similarities and differences among the textbooks regarding two constructs: first, the presentation of addition and subtraction of fractions, examined through the structure, selection and sequencing of topics, as well as the worked-out examples and representations employed; and second, the textbook expectations manifested in the cognitive demands of the tasks on adding and subtracting fractions and the type of the work required from students when considering these tasks.

INTRODUCTION

Textbook analysis, a relatively unexplored field (Johansson, 2003), appears to be increasingly attracting the interest of the research community (Pepin, Haggarty, & Keynes, 2001; Mesa, 2004; Rezat, 2006). In the last two decades researchers have expressed contrasting views as to what one can learn from analyzing mathematics textbooks. Some researchers have gone as far as to claim that textbook analysis can explain the differences in students' performance in international comparative studies (Fuson, Stigler, & Bartsch, 1988; Li, 2000). Other researchers have argued that textbooks exert little influence on instruction and on what students learn (Freeman & Porter, 1989). A more balanced viewpoint posits that textbooks afford *probabilistic* rather than deterministic opportunities to learn mathematics (Mesa, 2004; Valverde, Bianchi, Wolfe, Schmidt, & Houang 2002). Although this perspective acknowledges that textbook analysis is limited to portraying the *intended* and not the implemented curriculum, it holds that important insights can be gleaned from studying textbooks used in different countries by illustrating similarities and differences in the opportunities to learn mathematics offered to students around the world.

Studies show significant cross-national differences in the textbooks. Two large-scale studies that compared and contrasted the textbooks adopted in almost 40 countries concluded that textbooks vary "in a myriad of ways" (Schmidt, McKnight, Valverde, et al., 1997, p. 22) and that "they exhibit substantial differences in presenting and structuring pedagogical situations" (Valverde et al., 2002, p 17). Small-scale studies also found major differences between the mathematics textbooks used in China, Japan, the former Soviet Union, Taiwan, and the United States (e.g., Fuson, et al., 1988; Mayer, Sims, & Tajika, 1995) or between the textbooks used in different European

countries (Pepin et al., 2001). These studies have pursued different textbook-analysis approaches. Some have examined the textbooks as *a whole*, focusing on general textbook characteristics (e.g., physical appearance, the organization of the content across the book). Critics have argued that this approach, which we call a *horizontal analysis* of textbooks, fails to illuminate *substantial* differences in the learning opportunities offered to students in different countries, since topics are not treated in the same manner and with the same degree of emphasis in different textbooks (Howson, 1995). Other studies have moved into greater depth, attending to the ways in which textbooks treat a *single* mathematical concept (Fuson et al., 1988; Li, 2000, Mesa, 2004); we consider this approach a *vertical analysis* of textbooks. This approach, however, does not allow situating the concept under investigation within the broader context of the textbook. The studies that analyzed textbooks pursuing both a horizontal and a vertical analysis illustrate that such a complementary approach is feasible and worthwhile (Howson, 1995; Pepin et al., 2001). However, the field seems to be lacking a single coherent, useable framework that would allow for systematic fine-grained analyses of textbooks.

In the present paper, we report findings from a larger project in which we developed such a framework and applied it to study the treatment of fractions in the mathematics textbooks used in three countries, Cyprus, Ireland, and Taiwan. We focused on fractions, because teaching and learning fractions has traditionally been considered one of the most problematic areas in elementary school mathematics (Lamon, 1999). In this paper we scrutinize, in particular, the treatment of the addition and subtraction of fractions in these textbooks. As Verschaffel, Greer, and Torbeyns (2006, p. 65) noted in their review of the studies published in the last 15 PME proceedings, little work appeared in the area of the four operations on fractions, and specifically on the addition and subtraction of fractions, an area where even preservice teachers lack conceptual proficiency (Philippou & Christou, 1994). We analyzed textbooks from the aforementioned countries, because the existence of a national curriculum in each country facilitated the selection of textbooks. Notwithstanding this, the three countries have significant differences in history, size, language, economy, culture, and student attainment in international comparative studies. In addition, our experiences as students and teachers of elementary mathematics in these countries provided us with the contextual knowledge conducive to such an analysis.

The goal of our inquiry was twofold. First, we sought to investigate the similarities and differences in the *presentation* of the addition and subtraction of fractions in the textbooks of these countries. Second, by analyzing the tasks used in these textbooks, we aimed to explore the *expectations* that the textbook authors appear to hold for students when working in this area. By narrowing our attention to the addition and subtraction of fractions, we aimed for a more detailed analysis of the two foci under consideration, which we considered critical for shaping students' probabilistic learning opportunities.

METHODS

In developing the two-layer analytic framework used in the larger project (Figure 1), we pursued an iterative approach that combined results from the literature with a preliminary exploration of the textbooks used in the three countries. The literature review suggested criteria employed in previous cross-cultural textbook studies; we filtered and synthesized these criteria considering the particular textbooks to hand.

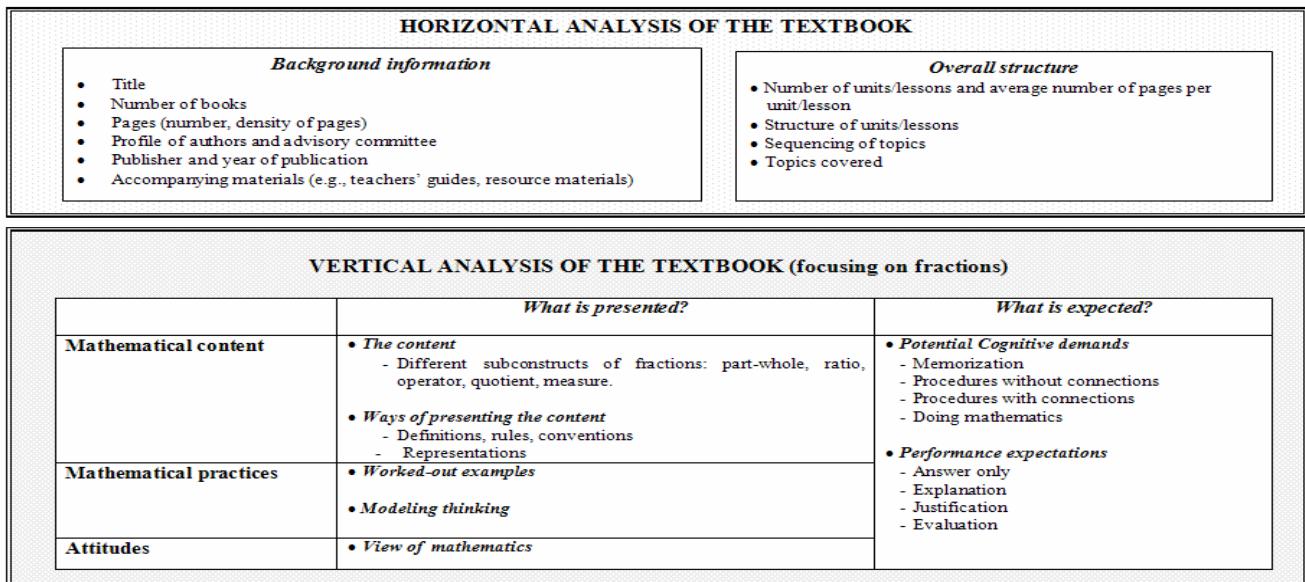


Figure 1. Framework used to analyze the mathematics textbooks.

For the purposes of the present paper, we analyzed (1) the structure, sequencing, and the topics covered with respect to the addition and subtraction of fractions (from the horizontal analysis); (2) the different constructs of fractions (cf., Behr, Lesh, Post, & Silver, 1983; Lamon, 1999); (3) the representations; (4) the worked-out examples employed in the textbooks (from the “what is presented” aspect of the vertical analysis); (5) the potential cognitive demands;¹ and (6) the performance expectations (from the “what is expected” aspect of the vertical analysis). We used the first four dimensions to analyze the *presentation* of the addition and subtraction of fractions and the latter two aspects to analyze the textbook *expectations*.

We analyzed five textbook series: the fourth-grade Cypriot textbook series used in all public schools in the country (Ministry of Education and Culture, 1998), two widely used Irish fifth-grade textbook series (Barry, Manning, O’Neill, & Roche, 2003; Courtney, 2002) and two fourth-grade textbook series widely used in Taiwan (Li & Huang, 2005; Yang & Miao, 2003). The disparity in the grades was due to the different

¹ We analyzed the potential cognitive demands using the task-analysis guide (Stein, Smith, Henningsen, & Silver, 2000, pp. 16-21), according to which *memorization* and *procedures-without-connections* tasks are considered intellectually undemanding and *procedures-with-connections* and *doing-mathematics* tasks are intellectually demanding. We also distinguished the tasks that require connecting the addition and subtraction of fractions to its underlying meaning but expect students to apply a *well-established* procedure from those that connect the procedure to its meaning, but the procedure required is not *well-established*.

grade-placement of the addition and subtraction of fractions in the three countries. The content of the Cypriot and the Taiwanese textbooks was translated into English to support the cooperative analysis of the content of all three textbooks by all four authors and the resolution of the disagreements that emerged.

FINDINGS

Presentation: Structure, Sequencing, and Topics Covered.

We found notable differences among the three countries. First, in Cyprus and Taiwan the addition and subtraction of fractions and mixed numbers with like denominators is formally introduced in the fourth grade; the additive operations on fractions and mixed numbers with unlike denominators are considered in fifth grade. In contrast, both Irish textbooks introduce the addition and subtraction of fractions and mixed numbers with similar or unlike denominators in fifth grade; no reference is made to adding or subtracting fractions in the fourth-grade textbooks.

Second, the Irish textbooks follow a slightly different sequence in the presentation of the content compared to the textbooks used in the other two countries: they first introduce the addition and subtraction of fractions with similar or unlike denominators, next they consider additive operations on mixed numbers whose fractional part has the same or different denominator, and finally, they offer students opportunities to solve word problems on the addition of fractions or mixed numbers. In contrast, the Cypriot and the Taiwanese textbooks first introduce the addition and subtraction of fractions with similar denominators and then move to the addition and subtraction of mixed numbers whose fractional part has the same denominator. In both countries, exercises that purport to help students develop procedural fluency are interwoven with word problems on the addition and subtraction of fractions.

Third, the textbooks in Ireland and Taiwan connect the addition of fractions with the multiplication of fractions as repeated addition; in the Cypriot textbooks, this connection is postponed until fifth grade, when the multiplication of fractions is formally introduced. In addition, the Taiwanese textbooks build connections between the addition of fractions, equivalent fractions, and the division of fractions as repeated subtraction by asking students to consider problems such as: “One bag has 100 small paper plates. 83 of 1/100 bag of plates equals how many 1/10 bags of paper plates? How many 1/100 bag of paper plates?” (Li & Huang, 2005, vol., 1, p. 11).

Presentation: Subconstructs of Fractions Presented in the Textbooks.

The *part-whole* interpretation of fractions is the dominant subconstruct used in all textbook series. The *measure* subconstruct, which is associated with the measure assigned to some interval appears less frequently; this interpretation is weakly reinforced, since in all cases the measure assigned to the interval(s) under consideration (e.g., the length of a line segment, the volume of the liquid in volumetric glasses) is already given (i.e., students are not expected to *measure* the interval). The remaining three subconstructs (i.e., ratio, operator, and quotient) are not present in any of the textbooks analyzed.

Presentation: Representations Employed in the Textbooks.

The representations in the Irish textbooks consist of mainly pre-partitioned circular units. The representations accompanying all worked-out examples and the tasks in the Cypriot textbooks are mostly circular or rectangular area representations, which, in all but one case (vol. 4, p. 46), are pre-partitioned. In contrast, the Taiwanese textbooks, in addition to the circular and rectangular area pre-partitioned representations, include linear representations (e.g., ropes) and volumetric representations (e.g., bottles of liquids). Besides these continuous models, the Taiwanese textbooks also include representations of discrete sets (e.g., bag of cookies or eggs, boxes of stationery materials or fruits). No textbook series employs a number line in presenting the addition and subtraction of fractions and mixed numbers.

Presentation: Worked-out Examples.

We found different approaches for scaffolding student understanding through worked-out examples; we illustrate these with the subtraction of mixed numbers (Figure 2).

Textbook A (Cypriot): Constantinos's father bought 3 pizzas for his son's birthday. The kids ate $1\frac{1}{3}$ pizzas. How many pizzas were left? Color the pizzas that were eaten. There were $2\frac{1}{4}$ apple pies in the pan. The kids ate $1\frac{1}{4}$ apple pies. How many apple pies were left? Color the apple pies that were eaten.

Textbook B (Taiwanese): How many less meters of colorful belts did Gi-Wen use compared to Wen-Ting? Write the corresponding mathematical expression and then find the answer. 2 minus 1 is 1. We cannot subtract $\frac{4}{5}$ from $\frac{3}{5}$. What should we do? $2\frac{3}{5} - 1\frac{4}{5} = (\quad)$ I converted all the mixed fractions into improper fractions. I converted $2\frac{3}{5}$ to $1\frac{8}{5}$.

Textbook C (Irish): From $3\frac{1}{5}$ take away $1\frac{3}{10}$. Subtract $2\frac{12}{10} - 1\frac{3}{10} = 1\frac{9}{10}$.

Textbook D (Irish): Example 2: $4\frac{1}{6} - 2\frac{3}{4}$. $\frac{1}{6} = \frac{2}{12}$, $\frac{3}{4} = \frac{9}{12}$. $4\frac{1}{6} = 4\frac{2}{12} = 3\frac{14}{12}$ (by renaming). $- 2\frac{3}{4} = - 2\frac{9}{12} = - 1\frac{5}{12}$.

Figure 2. Worked-out examples on the subtraction of mixed numbers (A: Cypriot textbook; B: Taiwanese textbook; C and D: Irish textbooks)

First, the Taiwanese worked-out example suggests two different ways of subtracting mixed numbers, the Irish textbooks suggest only one such way, and the Cypriot textbook does not suggest an algorithm. Second, whereas the Cypriot and the Taiwanese textbooks situate the worked-out example within a particular context, in both Irish textbooks the worked-out examples are not contextualized. Third, the Cypriot textbook uses two different part-whole representations; the Irish textbooks employ at most one. The representation used in the Taiwanese textbooks is closer to

the measure subconstruct of fractions. Fourth, the worked-out example in the Taiwanese textbooks is based on the *comparison* interpretation of subtraction, which is considered more complex than the *take-away* interpretation employed in the examples used in the Cypriot and one of the Irish textbooks.

Expectations: Potential Cognitive Demands.

From the 46 tasks of the Cypriot textbook series, 14 require employing adding and subtracting fractions *without connections to its meaning*. The remaining 32 tasks were classified as *procedures with connections to meaning*, but only 13 expect students to employ a procedure that is not modeled in the worked-out examples. The *procedures-without-connections* tasks are longer (e.g., in one such task students are expected to solve up to 24 similar exercises). A similar pattern was found in the Taiwanese textbooks. From the 44 tasks in the *Yang and Miao* series, 20 were classified as *procedures without connections*, and 24 as *procedures-with-connection to meaning*. Only 4 of the latter tasks require a procedure somewhat different from that modeled in the worked-out examples. Of the 67 tasks of the *Li and Huang* series, 12 were coded as *procedures without connections* and 55 as *procedures with connection to meaning*; yet, only 11 of these tasks require using a procedure that was not well-established. In the Irish *Barry et al.* series, 51 tasks were classified as *procedures without connections* and 9 tasks as *procedures with connections* that require using a well-established procedure. Nine tasks in the *Courtney* series were classified as *procedures without connections* – some of them have as many as 16 sub-parts – and six tasks were classified as *procedures with connections* that expect students to apply a well-established procedure. None of the tasks in the textbooks analyzed was classified as a *doing-mathematics*.

Tasks in the Taiwanese textbooks coded as *procedures-without-connections* are more complex. For example, besides asking students to find the sum or the difference of two fractions or mixed numbers, there are tasks asking students to find the missing addend or minuend/subtrahend (e.g., $8 \frac{5}{12} + ? = 10 \frac{1}{12}$; $? + 2 \frac{6}{7} = 5 \frac{3}{7}$; $20 \frac{3}{16} - ? = 7 \frac{9}{16}$; $? - 1 \frac{23}{30} = 3 \frac{19}{30}$ (Li & Huang, vol. 2, p. 23 & 25). Such tasks do not appear in the Irish textbooks; in the Cypriot textbook there are two such tasks, one that concerns operations within the unit (e.g., $? + 1/5 = 4/5$ and $3/7 - ? = 2/7$; vol. 3, p. 47) and the other on mixed numbers ($4 \frac{9}{12} + 1 \frac{1}{12} + ? = 5 \frac{3}{12} + 7$; vol. 4, p. 49).

Expectations: Performance Expectations.

All the tasks included in the Cypriot and the Irish textbooks ask students to provide a single answer. Whereas finding an answer is also the predominant performance expectation in the Taiwanese textbooks, these textbooks include several tasks in which students are asked to explain their solution approach: “The length of the red rope is 2 $\frac{37}{100}$ meters. The length of the white rope is $\frac{14}{100}$ meters. How long are the two ropes altogether? Write the calculation expression for this problem and give the final result in the form of a mixed number. Explain your thinking process in solving this problem” (Li & Huang, vol.1, p. 47).

DISCUSSION

The framework employed in the present study in analyzing the Cypriot, Irish, and Taiwanese textbooks helped identify key similarities and differences in the presentation and textbook expectations with respect to the addition and subtraction of fractions. For instance, the part-whole subconstruct of fractions was found to be dominant in the textbooks of all three countries; in presenting this interpretation of fractions the textbooks mainly used pre-partitioned area models. Additionally, the textbooks in all three countries introduced the addition and the subtraction of fractions and then shifted to the addition and subtraction of mixed numbers. On the other hand, our analysis pointed to some notable differences between the Taiwanese textbooks and those used in Cyprus and Ireland. In particular, the Taiwanese textbooks employ a greater variety of representations compared to the Cypriot and Irish textbooks; they use a greater variety of representations (linear, area, volumetric, and discrete sets representations), which is considered critical for supporting students' understanding of rational numbers (Lamon, 1999). The Taiwanese textbooks also build more connections of the addition and subtraction to the multiplication and division of fractions; moreover, they employ more complex situations in their worked-out examples and explicitly model more than one way to carry out an algorithm.

To what extent can the differences between the Taiwanese textbooks, on the one hand, and the Cypriot and Irish textbooks, on the other, explain the superior performance of the Taiwanese students on international comparative studies? Does the dominant role of the part-whole interpretation of fractions suggest that the association, found in a previous study (Charalambous & Pitta-Pantazi, 2005) between this subconstruct and the additive operations of fractions, is mainly an artifact of the curriculum? What might be a more appropriate way of presenting a worked-out example? Although the present study does not provide answers to these questions, it illustrates the affordances of a more fine-grained textbook analysis facilitated by our framework in generating hypotheses that are worth further investigation, and which might help better understand the connections between the intended, the implemented, and the attained curriculum.

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THE DEVELOPMENT OF PRIMARY STUDENTS' KNOWLEDGE OF THE STRUCTURED NUMBER LINE

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This paper investigates how primary students' knowledge of the structured number line develops over time. Effective number line use is impacted by an understanding of this graphic as a representation that encodes information by the placement of marks on an axis and an appreciation that these marks are representations of length rather than labelled points. Generally, the results revealed development of number line knowledge over time. However, although correlations between number line items were positive, they were only moderate or low. Additionally, there was low shared variance between items even though students solved the same items in three consecutive years. Noteworthy were the statistically significant gender differences in favour of boys over the 3-year-period – findings that warrants further investigation.

INTRODUCTION

Structured number lines are often used as instructional aids, in texts, and on tests in the primary years. These number lines are distinguished by the presence of marked line segments – which are absent from empty number lines (Klein, Beishuizen, & Treffers, 1998). Advocates of structured number lines argue that they have various benefits for students, for example in number sequencing (e.g., Wiegel, 1998) and for concretising operations (e.g., Davis & Simmt, 2003). In contrast, others have reported that structured number lines are neither effective as conceptual supports (e.g., Fuson, Smith, & Cicero, 1997) nor are number line items valid measures of rational number knowledge (e.g., Ni, 2000). This disagreement in the literature about the utility of number lines is problematic because it fails to provide adequate guidance for educators who are trying to improve student outcomes. Thus, further research on number lines is essential if they are to be effective conceptual tools for students. Understanding the role of tools in developing mathematical proficiency is fundamental to achieving equity in student outcomes (Ball, 2004).

Our previous studies of primary students' knowledge and use of structured number lines (Diezmann & Lowrie, 2006; Lowrie & Diezmann, 2005, in press) have provided some insights. However, like much of the literature on number line use, our earlier studies generally examined students' knowledge at a single point in time or over a single year. This paper extends on our past work by exploring how students' knowledge of the structured number line develops over a 3-year-period.

THE STRUCTURED NUMBER LINE

The structured number line has potential cognitive advantages for understanding various aspects of mathematics. However, underpinning the actual benefit from these advantages are key understandings about the structured number line.

Advantages of Number Line Use

Cognitively, number lines have three potential advantages for users. The first two advantages relate to *mathematical variability* and *perceptual variability* (Dienes, 1964). Mathematical variability is demonstrated on the number line by its use as a generic representation or tool which can show many mathematical concepts including the position of a fraction in a number sequence and the continuity of rational numbers. Perceptual variability in mathematics is illustrated by the number line when it is used as one of a variety of representations to show different aspects of the same concept. For example, a half can be represented on a number line, on a pie graph, and on an array. The final cognitive advantage of the number line is *representational transfer* (Novick, 1990). This process of knowing how to use a representation for a routine task can cue the user about how to use it on a novel task. For example, knowing how to identify a missing whole number on a number line might cue students about how to find a missing decimal number on another number line. Despite the potential cognitive advantages, many primary students do not use the number line effectively (Diezmann & Lowrie, 2006; Lowrie & Diezmann, 2005).

Key Understandings about the Number Line

There are two key understandings that students need to develop about the structured number line. The first understanding relates to the number line as a graphic. Structured number lines are part of the increasing number of graphics that are used for the management, communication, and analysis of information (Harris, 1996). Number lines are part of the *Axis* graphic language that uses a single position to encode information by the placement of a mark on an axis (See Appendix for examples of number line items). There are five other graphic languages that use distinct perceptual elements and encoding techniques (Mackinlay, 1999): *Opposed-position* (e.g., bar graph), *Retinal-list* (e.g., saturation graphics such as population density), *Map*, *Connection* (e.g., family tree), and *Miscellaneous* (e.g., calendar). See Lowrie and Diezmann (2005) for a description of these languages and examples. Although these six graphic languages differ perceptually, they are related. Our previous research on Axis Languages revealed positive and statistically significant correlations at a $p \leq .01$ level between students' performance and pairings with other graphical languages with the exception of the Axis-Opposed-position correlation ($r = .15$, $p \leq .05$) (Lowrie & Diezmann, 2005). Opposed-position languages consist of graphics where information is encoded by a marked set that is positioned between two axes (e.g., a bar graph). The weak correlation between Axis items and Opposed-position items was not surprising because these languages are structurally dissimilar with information encoded in one dimension two dimensions respectively.

The second understanding is that the structured number line is a measurement model. Structured number lines have marked line segments, and hence, numbers on the line are representations of length rather than simply labelled points (Fuson, 1984). Students' conceptualisation of the number line as a measurement model impacts directly on their success. For example, in interviews on number line items with 67 10- to 11-year-olds, we found that successful and unsuccessful students differed from each other in their use of the number line (Diezmann & Lowrie, 2006). Typically, successful students demonstrated a measurement orientation in their identification of unknown numbers represented by letters on a number line: "I chose D because B is (to the) right, *a bit far away* from 20 and C is *in the middle* and I thought that would be about 10 and A would be *too close* to the 0 to be 17" (emphasis added). In contrast, a common response from unsuccessful students was to solely employ counting to identify the unknown value: "I think it (the number 17) should go there (D) because it's next to 20 and it goes 19, 18 then 17 (emphasis added)". The use of a simple counting strategy is inappropriate because it incorrectly assumes that (1) the marked line segments are evenly spaced and that (2) the distance between each segment represents one unit. As the spacing between markings of line segments is variable on structured number lines and the distance between the segments can represent any number of units, students who solely use a counting strategy are unlikely to be successful. Although counting alone is an inappropriate strategy for use on the structured number line, it can be appropriate for the empty number line (e.g., Klein et al., 1998).

DESIGN AND METHODS

This investigation focuses on students' performance on number line items over a 3-year period. The longitudinal study employed the repeated collection of mass testing data over three annual time intervals. This design allowed for an examination of the magnitude and direction of change in students' knowledge of number lines. This study is part of a larger study (Diezmann & Lowrie, 2006; Lowrie & Diezmann, 2005, in press) in which we are monitoring the development of primary students' ability to decode the six types of graphical languages including Axis languages of which number lines are a key exemplar. Here, we report solely on the longitudinal data for the number line items. The research questions were:

1. How does students' performance on number line items vary over time?
2. What are the relationships in students' performance across number line items over a 3-year-period?
3. Were there are gender differences in students' decoding performance on number line items over time?

The Participants

A total of 328 participants ($M = 204$; $F = 174$) commenced in the study when they were 10 to 11 year-olds (Grade 4 in New South Wales; Grade 5 in Queensland) and completed the three annual tests in consecutive years. These students were drawn from

six primary schools across two states and included students from state, church-based, and independent schools. The socio-economic status of the students was varied and less than 5% of students had English as a second language.

The Instrument

The Graphical Languages in Mathematics [GLIM] Test is a 36-item multiple choice test that was designed to investigate students' performance on each of six graphical languages (See Lowrie & Diezmann, 2005). This instrument comprises six items that are graduated in difficulty for each of the graphic languages. The four number line items used in this study (See Appendix) were drawn from the Axis sub-test. In reporting the results, the Axis items are denoted by an item number followed by the year of the study. Hence, *Axis 2/3* refers to Item 2 in the third year of the study.

Data Collection and Analysis

Data on students' performance on the GLIM test was collected through mass testing of the same items in three consecutive years. Students' performance on these tasks was scored as "1" for a correct response and "0" for an incorrect response. Correlations were generated from the data to investigate the relationships across number line items (Research Question 1). Multivariate and univariate statistics were employed to examine students' performance on number line items over time (Research Question 2) and to explore gender and item success (Research Question 3).

RESULTS AND DISCUSSION

1. How does students' performance on number line items vary over time?

A MANOVA showed a statistically significant difference between year and item success over the 3-year period [$F(8, 2086) = 11.4, p \leq .001$]. Post-hoc analysis revealed statistically significant differences in the students' performances across three of the number line items over three years of schooling [*Axis 2* ($F(2, 1052) = 7.1, p \leq .001$); *Axis 3* ($F(2, 1052) = 37.1, p \leq .001$); and *Axis 4* ($F(2, 1052) = 9.1, p \leq .001$) (See Table 1). Student performance on Item 1 was not statistically different over time [*Axis 1* ($F(2, 1052) = 2.5, p > .08$)], however, this was due in part to a ceiling effect. The most dramatic improvement in student performance was on Item 3 (from 37% to 67%). This item required students to establish the time taken for the second of four stages in the life cycle of a butterfly. The correct response required students to identify the starting and finishing points of the relevant time period, and to calculate how many days in the period (See Appendix). The marking of the line segments in two day rather than one day periods added to the complexity of the item.

2. What are the relationships in performance across number line items over a 3-year-period?

Correlations were generated for student responses across the first year and third year of the study for each of the four number line items. These four items were positively correlated with each other [*Axis1/1-Axis1/3* ($r = .34, p \leq .001$); *Axis2/1-Axis2/3* (r

$= .08, p >.05$); *Axis3/1-Axis3/3* ($r = .35, p \le .001$); and *Axis4/1-Axis4/3* ($r = .20, p \le .001$]). Although three of the correlations had strong statistical significance, the items were only weakly or moderately correlated with each other. Previously, we found moderate correlations among the six graphical languages (Lowrie & Diezmann, in press); however, the results of the present study are surprising since relationships were weak between identical items within the axis language over time. Moreover, although the students solved these items in each of three consecutive years, the greatest shared variance between the items [*Axis3/1-Axis3/3*] was only 12%.

		YEAR 1	YEAR 2	YEAR 3
Item 1	Total	.88(.33)	89(.31)	.93(.26)
	Male	.91(.28)	.91(.29)	.95(.31)
	Female	.83(.38)	.87(.34)	.90(.30)
Item 2	Total	.79(.41)	.85(.36)	.89(.31)
	Male	.82(.38)	.87(.33)	.90(.30)
	Female	.75(.44)	.82(.38)	.88(.33)
Item 3	Total	.37(.48)	.58(.50)	.67(.47)
	Male	.44(.50)	.67(.42)	.73(.44)
	Female	.28(.45)	.47(.50)	.61 (.49)
Item 4	Total	.31(.46)	.41(.49)	.46(.50)
	Male	.39(.49)	.50(.50)	.55(.50)
	Female	.21(.41)	.29(.46)	.36(.82)

Table 1: Means and standard deviations for number line items (and gender).

3. Were there are gender differences in students' decoding performance on number line items over time?

The third aim of this investigation was to establish whether there were gender differences in students' decoding performance on the number line items. The mean scores for male students were higher than for female students in all four categories across the three years of the study (See Table 1). A MANOVA showed a statistically significant difference between gender and item success over the 3-year period [$F(4, 1044) = 16.6, p \le .001$]. ANOVAs revealed statistically significant differences across the gender variable for each item [*Axis 1* ($F(1, 1052) = 9.3, p \le .001$); *Axis 2* ($F(1,$

$1052) = 5.2, p \leq .02$); *Axis 3* ($F(1, 1052) = 29.7, p \leq .001$); and *Axis 4* ($F(1, 1052) = 42.8, p \leq .001$). These gender differences are consistent with other research. Previously, we found statistically significant gender differences in favour of males at Grade 4 and Grade 5 on a set of six Axis Language tasks (Lowrie & Diezmann, in press). Hannula (2003) also reported that boys outperformed girls on number line tasks in a Finnish study of Grade 5 ($n = 1154$) and Grade 7 ($n = 1525$) students. Thus, gender appears to be a key variable in students' success on the number line.

CONCLUSION

This study revealed three points of interest. First, the moderate or low correlations over the 3-year-period between number line items and the low shared variance indicate that students perceived these items to be dissimilar rather than similar. One plausible explanation for this failure to detect similarity is that students were paying attention to the surface detail rather than the structure of the graphic. The number line items are similar at a structural level in that each item makes use of the placement of mark on an axis (McKinlay, 1999). However, they are dissimilar at a surface level, for example in the context, the range of numbers, and the visual presentation of the graphic. A focus on the surface detail rather than the structure limits the possibility of representational transfer where knowledge of a particular representation is transferred to another representation (Novick, 1990). This finding indicates the need for an emphasis on perceptual variability (Dienes, 1964) in number line examples.

Second, the longitudinal results indicate that there can be a dramatic increase in students' performance over a couple of years. It is unlikely that this increase was solely due to students' improved understanding of the number line as a graphic because (1) the curriculum did not include explicit instruction about number lines, and (2) in addition to knowledge of the graphic, success on graphically-oriented items requires a simultaneous consideration of knowledge of mathematical content and context and an adequate level of literacy to comprehend the text (Lowrie & Diezmann, in press). It seems more likely that with additional experience and schooling, students had mastered the mathematical, contextual and/or literacy elements that had previously been "stumbling blocks" to success.

Third, the finding of gender differences is of particular interest because it challenges Voyer, Voyer, and Bryden (1995)'s conclusion from a meta-analysis that the only gender difference in spatially-oriented tasks in under 13-year-old students is limited to performance on mental rotation tests. In this study, we found gender differences within and across primary-aged students at three grade levels. Thus, gender differences in the latter years of primary schooling warrant further investigation.

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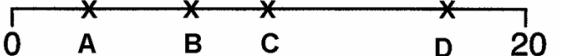
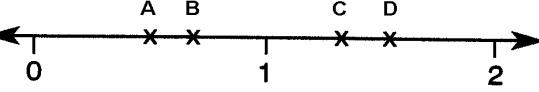
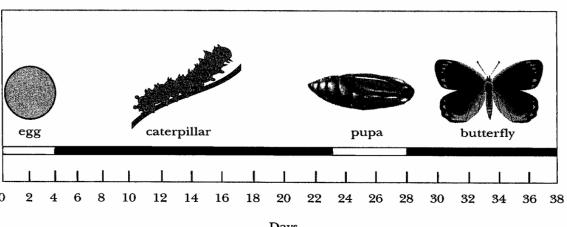
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APPENDIX: NUMBER LINE ITEMS

<p>1. Estimate where you think 17 should go on this number line.</p>  <p>0 A B C D 20</p> <p>(Adapted from Queensland Studies Authority, 2000a, p. 11)</p>	<p>2. Estimate where you think 1.3 should go on this number line.</p>  <p>0 A B C D 1 2</p> <p>(Adapted from Queensland Studies Authority, 2000b, p. 8)</p>
<p>3. The following graph shows the length of time taken for the four stages in the life of a butterfly. How many days are there in the caterpillar stage?</p>  <p>0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 Days</p> <p>egg caterpillar pupa butterfly</p> <p>(Educational Testing Centre, 2001, p. 2)</p>	<p>Bay City Exton Yardville</p>  <p>4. On the road shown above, the distance from Bay City to Exton is 60 kilometres. What is the distance from Bay City to Yardville?</p> <p>(National Centre for Educational Statistics, 2003)</p>

REASONING WITH METAPHORS AND CONSTRUCTING AN UNDERSTANDING OF THE MATHEMATICAL FUNCTION CONCEPT

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The paper describes the nature of two intermediate algebra students' use of metaphors in constructing an understanding of the mathematical function concept, and in reasoning with them while addressing mathematics questions. The interview analysis of the two students indicated three categories of metaphor use. Furthermore, these metaphors appeared to have encouraged the formation of some of the misconceptions reported in literature.

INTRODUCTION

The concept of function is essential to college mathematics learning, especially for those covered in courses calculus and up. Because of the importance of function in higher mathematical understanding, it is necessary to make sure that students gain an accurate understanding of the basics of the concept before taking advanced courses. Many students in entry-level courses do not have a background of mathematical knowledge from which to attempt to make sense of newly introduced mathematical information (Sfard, 1997). Thus they turn to their existing knowledge, which mainly originates from everyday experiences. This paper reports on the metaphors originating from everyday experiences intermediate algebra students used to construct an understanding of the mathematical function concept.

Metaphor

A metaphor can be defined as an implicit analogy (Presmeg, 1997). Presmeg adds that a metaphor has both ground and tension. According to her, similarities between concepts constitute the ground and differences constitute the tension. For instance, for the mathematical statement, "A is an open set," the tension of a metaphor may be the physical idea of openness (an open space view) without a boundary and the mathematical idea of an open set with a boundary. Consider a set of all points that satisfy the inequality, $x^2+y^2<1$. Here, the open set is bounded by the unit circle. Considering that similarities between concepts (source and target) are mainly determined by students based on past experiences, rather than being given to them (Sfard, 1997), students, in this case, may apply the no boundary characteristic of the source concept, and come to a conclusion that the particular set is not open since it has a boundary. Moreover, the students' misconception may further be strengthened if initially introduced examples of mathematical open sets are those without boundaries.

Metaphors play an important role in reasoning in mathematics (Lakoff and Nunez, 2000; Presmeg, 1997). Presmeg illustrates how one may reason with metaphors in her example of a high school student who used “Dome” metaphor to reason in solving the question of finding the sum of the first 30 terms of a sequence (5, 8,11,...). This high school student’s personal metaphor seemed to play a significant role in solving the problem. The students whose interview excerpts are reported here also appeared to reason with their idiosyncratic metaphors in responding to mathematics questions.

Function

Studies on the function concept focused mainly on students’ conceptualizations (Carlson, 1998; Hansson and Grevholm, 2003; Selden and Selden, 1996; Vinner, 1990; Williams, 1998). Some of the conceptualizations (misconceptions) reported are: 1) Function is an algebraic term/a formula/an equation; 2) Functions should be given by one rule; 3) Graphs of functions should be regular and systemic; 4) The constant algebraic form, $y=c$, c is a *constant*, is not considered as the representation of a function. The findings of our study indicate that some of these conceptions may have been the result of a metaphorical reasoning while attempting to make sense of mathematical information. Many studies on the conceptualizations used instruments such as paper and pencil tests, which do not reveal metaphorical reasoning. Others on the other hand used instruments that may have revealed reasoning with metaphors. They however may not have reported the cases because of the differing focus of their investigation. We report student responses that appeared irrelevant to the focus of our study at first, but revealed meaning and relevance during interviews.

METHODOLOGY

Data reported is based on two intermediate algebra students’ interviews conducted as part of a study that gathered concept maps (Novak and Gowin, 1984) and interpretive essays along with interviews to investigate students’ conceptualizations as they were introduced to the various aspects of the mathematical function concept through out a semester in Fall 2003 from two intermediate algebra sections at a four year midsize Southwest University in the United States. The intermediate algebra course at the University provides basic mathematical concepts and skills that are prerequisite for college mathematics courses such as pre-calculus and calculus. Both sections of the course were traditional in the sense that its material was covered through a lecture by instructor mode. Among the three students who volunteered to be interviewed, two displayed reasoning with metaphors originating from everyday experiences. We refer to one of the two as student L (American female with an intended major in criminology), and the other as student Y (Hispanic-American female with an intended major in bilingual education). It should also be noted that student Y’s instructor used “blender” analogy to introduce the mathematical function concept.

Each interview was videotaped and lasted about an hour and a half. Interviews began with more personal questions, and proceeded to questions on the function concept and concept maps. Students' inaccurate responses were not corrected through out the interviews in order to further understand the sources of their mistakes, and to eliminate the possible influence of the interviewer's opinion on students' thought processes. Two mathematics graduate students transcribed the interviews, which were then analyzed by the same graduate students and the investigator in order to identify emerging patterns.

RESULTS

Metaphors and Students' Descriptions

During the interviews, students provided descriptions for their understanding of the function concept in both everyday and mathematical settings. The descriptions came about when students were asked to talk about the terms displayed on their concept maps; to talk about what they think of when they hear the word "function;" and to address the question, "what a function is." What follows are the excerpts from the interviews providing information on the students' descriptions.

Student L's Description

When student L was asked to tell the first thing she thought of right after hearing the word "function," she replied "relations." She however did not add anything further in support of what she meant by relations. Later in the interview, she was prompted to come up with an application of a function. With this question, the interviewer's intent was to investigate whether the student displayed any knowledge of an application specifically in the context of mathematics. The student's response was:

L: Well, in terms of math or in terms of?

Interviewer: In general.

L: Well, something functiony would be, something general is **running the way that it is supposed to**. Or if you are trying to plan a function, it would be something that would help something to **flow or to run properly**. That's what I would think about the word function.

When asked how she would describe the concept in the context of mathematics, she explained:

L: In mathematical setting, function would be something that where all of the numbers plugged in ..all of the numbers that were plugged in would create mmm something that was **consistent** like a graph it would be consistent.

Interviewer: Elaborate on consistent.

L: Consistent I think meaning like ..like if I were to be plugging in points on a graph in order to get a straight line if I if there were some points that I plotted that would not allow me to draw the straight line then then the points that I have, that I was given would not be consistent with the graph. Whereas if I had ordered pairs and then I plotted them and they did make the straight line then they would be consistent with the graph...

On this excerpt, she appears to consider the term “*consistent*” as obtaining what is expected. One may notice the similarities between her use of the phrase “*running the way that it is supposed to*” in describing the term (function) from everyday experiences, and the use of “*consistent*” in mathematical setting. In everyday setting, she uses the term as an entity that runs as expected, or helps run things properly. In mathematical setting, she thinks that a table of values represents a function if the expected outcome (a line in her case) is obtained as a result of plotting the table values. Notice that her two descriptions share a common notion of “*consistent with what is expected*.” L seems to consider this as a similarity between everyday and mathematical meanings of the concept. That is, for her, everyday meanings appear to become metaphors that she uses to construct an understanding for the mathematical function concept.

Student Y’s Description

Right after constructing her concept map, student Y was asked to talk about the terms and links displayed on her map (See figure 1 for the map). She replied:

Y: I basically see a function...as a **key** as an **answer** like problem like a **key** when you open the door. It is a **problem** and then function is the **answer**, a **resolution**, a **conclusion** to...

Here, she seems to hold an understanding of a function as being an entity that “*resolves*” problems. Considering the possibility that Y may have provided a description from everyday experiences with the concept, during her interview, she was further prompted to describe her understanding specifically in the context of mathematics. The following excerpt reveals her description in mathematical setting:

Y: Also like a key like you putting it in like what you do with function you put it in...

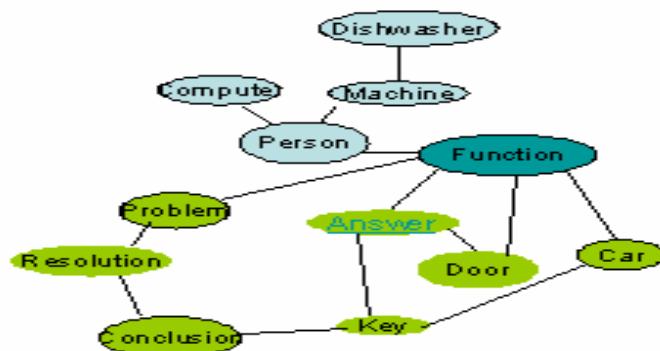


Figure 1. Student Y’s concept map (Redrawn).

In this excerpt, she appears to describe a function as a “*key*” or “*the role of a key*” to solve problems. Later in the interview, she was asked specifically what the term “*key*” meant to her. She stated that she considers “*key*” as the value of an independent variable of an equation with two variables, and that she uses the “*key*” to “*resolve*” a door/problem by putting the “*key*” in, and as a result opening the “*door*.” She seemed to be referring to an algebraic equation as representing a door to be resolved, and the

value of its variable as a key to be used to evaluate the equation. The following excerpt further explains what “key” meant to her in mathematics:

Y:...like that is the **key** [pointing to $x=2$ written on a paper] and you just sort of put it in there [pointing to $h(x)=5x+1$ written on a paper] to **resolve** it, to open the door with...

Over all, one can see that student Y’s descriptions in both contexts contain terms and phrases that are not only similar but many are the same.

Metaphorical Reasoning in Responses to Mathematics Questions

During the interviews, both students were given various representations, and asked to decide whether they formed functions. These representations included numerical, algebraic and graphical forms as well as non-examples of functions. What follows are the excerpts of students’ responses from the interviews, which shed light on the students’ use of metaphorical reasoning while answering the questions.

Student L’s Response

When student L was given a table of values and asked whether it forms a function, after plotting the points, she said “*no*,” and explained:

L: Because it didn’t it didn’t **flow harmoniously** like how this one did. Where it was just very simple I would just draw a line this would create some kind of problem for me because I will be able to draw the line and there will be my third point it would be somewhere off the line...



Figure 2 shows a section of the video clip where student L was explaining why she thought the table values did not represent a function. As seen on the figure, she was focusing on the plotted point seen at the tip of her line. She used the particular point to determine whether this was the expected

Figure 2. Video section from L’s interview

type of graph (a line for her case). She seemed to reason with her personal metaphor of an entity “*running the way that it is supposed to*” in responding to the particular question. Furthermore, her use of the phrase “*flow harmoniously*” indicates that she may have been considering the graphs of functions as regular/systemic. This is in agreement with similar findings reported in literature. Later in her interview when she was asked to give an example of a function, she gave the equation, “ $3x=9$,” as one. When asked whether we could prove that her example represents a function, she said:

L: If we were to substitute into x that would equal 9 Ya. I If I were to put 3 into 9 it equal, it **equal 9** but I did not write $x=3$...if I were to put $x=2$, it would not be a function because 3 times 2 is 6 that is not 9.

In this excerpt, L seems to be applying her metaphor of “*consistent with what is expected*” as meaning consistent with the expected result 9. What she appears to metaphorically think is that nine is what is expected as a result of a substitution, and if she gets 9 for the assigned value of x then her equation represents a function. If she

does not obtain 9 then “*it would not be a function.*” In short, throughout the interview student L consistently used her metaphors in her responses to mathematics questions.

Student Y’s Response

When student Y was given the algebraic expression, $h(x)=5x+1$, and asked whether it represents a function, she said “*Ya Ohh No.*” After a request for a further explanation, she stated: “*Not till you have x equaling something...*” Recall that student L also used a phrase similar to this one in her response regarding whether we could prove her example, “ $3x=9$,” forms a function. Student Y consistently used the term “*key*” to metaphorically describe the role of an independent variable in deciding whether an algebraic description determines a function. Without the presence of a numerical value of an independent variable, she was not willing to call an algebraic expression representing a function. Later in the interview, the interviewer also asked whether “ $f(x)=5, x=1$ ” (note that here, $x=1$ was included to eliminate a possible “no” answer) represents a function. Y’s response was: “*No, because no x over here* [pointing the expression, $f(x)=5$]...” Here, student Y appears not to consider these kinds of expressions as forming functions. This is also in agreement with the papers reporting a similar conceptualization among high school and college students. Her reasoning seems to have been influenced by the metaphors that she used in constructing an understanding of the concept in mathematics. That is, according to her metaphors, she needs a problem to solve which the constant expressions do not provide. She therefore rejects them as function representations. She also reasons metaphorically to answer a question of whether a table of values forms a function. She says: “*You need a problem for that...*” Y appears to consider a function as an entity that solves problems. She consistently used this notion in determining whether an expression represents a mathematical function. It is clear that she reasons with her “*Key and Door*” analogy in responding to mathematical questions.

DISCUSSION AND CONCLUSION

The students used three kinds of metaphors in their descriptions of the mathematical function concept, and in their responses to mathematics questions. Table 1 outlines the metaphors and the resulting conceptualizations. During the interviews, student L consistently used the first two metaphors to describe her understanding of mathematical function concept as an algebraic equation or a table of values that leads to an expected outcome when graphed. She expected to have linear graphs as a result of plotting points. She mainly used graphing as a mean to determine if a given form (algebraic expression and table) represents a function. Moreover, she metaphorically reasoned that the equations of one-unknowns are the representatives of functions only if the assigned values of the unknowns satisfy the equations (i.e., $3x=9$ is a function representation only when $x=3$). Similarly, student Y reasoned using the third metaphor shown in Table 1 to argue that equations with two variables (i.e., $y=3x+1$) represent functions only if the independent variables are assigned numerical values.

This furthermore prevented student L from considering tables of values and constant algebraic forms (i.e., $f(x)=c$) as function representations.

Table 1: Metaphors and Resulting Conceptualizations.

Source	Attributes	Target (Conceptualization)
Functionality of an object /Machine (Adjective).	“ <i>Functiony</i> ”- Running the way it is supposed to.	Equation or table of values whose graphs are lines. Equation with one unknown with an assigned numerical value satisfying the equation. Graph of a function is regular/systemic.
An entity that runs an event (noun).	To flow or to run properly.	The same as the first case.
Key and Door (noun)	Role of key-resolves problems.	Equation with two variables where the independent variable is assigned a numerical value. The constant algebraic form, $f(x)=c$, is not a function representation.

In summary, reasoning with the personal metaphors appeared to lead both students to the formation of some of the misconceptions reported in literature. They did not seem to consider equations as the representations of functions; rather, they seem to consider functions as equations. Both students seemed to lack the understanding of a function as a relationship. Furthermore, L’s metaphors seemed to have encouraged her to consider graphs of functions as regular/systemic. Similarly, Y’s metaphors appeared to allow the formation of an understanding that the constant algebraic forms do not represent functions.

Even though the findings reported here came from only two intermediate algebra students, the number (17 students out of 49) of intermediate algebra students who displayed everyday terms (such as machine, computer, and event) on their concept maps indicates that the phenomena observed on the two students’ interviews may not be unique. It may in fact be common among many students, especially among those enrolled in high school and early college mathematics courses. The findings reported here however can not be generalized based only on the two students’ cases. Further investigation, including more students with a variety of mathematics backgrounds, ethnicities, and languages spoken, is in order.

Recalling that student Y was from a section whose instructor used a “blender” analogy to introduce the mathematical function concept, one implication of the findings for the teaching and learning mathematics may be that the teachers of mathematics need to be cautious with the use of analogies such as “function machine” and “juice maker” in introducing the concept. As Max Black indicates, “*Similarity is created in the mind of conceivers of the metaphor rather than being given to them*” (reported in Sfard (1997, page 342)). What is tension in the eye of a teacher may become the ground for students. It is an unavoidable fact that students bring their everyday experiences into mathematics classroom, and consider them during the process of conceptualizing a newly acquired mathematical concept. Teachers may need to explicitly cover the relevant similarities between source and target concepts whenever there is a potential of applying irrelevant aspects by students. In the case of the function concept, the relationship aspect of the analogies used needs to be emphasized, and the irrelevant aspects need to be covered explicitly to make sure students do not consider these attributes while constructing an understanding.

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EXPLORING THE ENGLISH PROFICIENCY-MATHEMATICAL PROFICIENCY RELATIONSHIP IN LEARNERS: AN INVESTIGATION USING INSTRUCTIONAL ENGLISH COMPUTER SOFTWARE

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The difficulty of teaching and learning mathematics in a language that is not the learners' home language (e.g. English) is well documented. It can be argued that underachievement by South African learners in most rural schools is due to a lack of opportunity to participate in meaningful and challenging learning experience (sometimes due to lack of proficiency in English) rather than to a lack of ability or potential. This study investigated how improvement of learners' English language proficiency enables or constrains the development of mathematical proficiency. English Computer software was used as intervention to improve the English Language proficiency of 45 learners. Statistical methods were used to analyse the pre- and post-tests in order to compare these learners with learners from another class of 48. The classroom interaction in the mathematics class before and after the intervention was analysed in order to ascertain whether or not the mathematics interaction has been enabled or constrained. The findings of this study were that, first, any attempt to improve the language proficiency of learners with the aim of improving academic proficiency should be done in such a way as to develop concurrently, both the Basic interpersonal communicative skills and the cognitive-academic language proficiency; second, proficiency in the language of instruction (English) is an important index in mathematics proficiency, but improvement of learners' language proficiency, even though important for achievement in mathematics, may not be sufficient to impact on classroom interaction. The teacher's ability to draw on learner's linguistic resources is also of critical importance.

INTRODUCTION

Research study and philosophies dealing with the relationship between language proficiency and mathematical proficiency have either positioned the one as dependent on the other (Peal & Lampert, Cummins, 1978; Baker, 1988; 1962, Bialystock, 1992, in Lyon, 1996; Clarkson, 1992; Wales, 1977; Freitag, 1997; Holton, Anderson, Thomas, and Fletcher, 1999, in Albert, 2001; Taylor 2002) or the two as autonomous (Macnamara, 1977; Chomsky, 1975; Henney, in Aiken, 1972). In South Africa, even though the constitution gives provision for learners to learn in any of the 11 official languages of their choice, most learners learn mathematics in English which for most, is not their first or home language. Underachievement in Grade 12 mathematics examinations has been found to be more prevalent amongst learners who use the English language less frequently at home (Simkins in Taylor, Muller & Vinjevold, 2003) and in areas where English is less frequently used at home.

Most research dealing with language issues in mathematics education have documented that proficiency in the language of learning and teaching is important for mathematical proficiency¹ (e.g. Howie, 2002). But previous research into the relationship between English language proficiency and mathematics proficiency was not done in a classroom where there was an explicit attempt to improve learners' language proficiency using computer software. The study reported here investigated how the improvement of learners' English proficiency (using the English literacy computer software – ASTRALAB – designed to promote English proficiency) in one South African school, enabled or constrained the development of mathematical proficiency in learners. The study was organised to answer the broad question: Whether and how does improving learners' proficiency in English enable or constrain mathematical proficiency?

THEORETICAL ORIENTATION

Douady (1997) contends that to know mathematics involves a double aspect. It involves firstly the acquisition, at a functional level, certain concepts and theorems that can be used to solve problems and interpret information, and also be able to pose new questions_ (p. 374). Secondly, to know mathematics is to be able to identify concepts and theorems as elements of a scientifically and socially recognised corpus of knowledge. It is also to be able to formulate definitions, and to state theorems belonging to this corpus and to prove them_ (p. 375). What role does language play in the knowing of mathematics? Pirie (1998) and Driscoll (1983) contend that mathematics symbolism *is* the mathematics itself and language serves to interpret the mathematics symbol. In the relationship between language and mathematics, language serves as a medium through which mathematical ideas are expressed and shared (Brown, 1997; Setati, 2005). It can be argued, as Rotman, (1993, in Ernest, 1994: 38) does, that mathematics is an activity which uses written inscription and language to create, record and justify its knowledge_. Language, thus, plays an important role in the genesis, acquisition, communication, formulation and justification of mathematical knowledge – and indeed, knowledge in general (Ernest, 1994; Lerman, 2001).

It is with the above in mind that this study is informed by the socio-cultural theory of learning. The socio-cultural perspective proposes that learning is a social process and happens through participation in cultural practices (Doolittle, 1997). Learning, thus, involves becoming enculturated and enculturation into a community of practice in which a learner finds him/herself and it (learning) is marked by the use of conceptual tools like language. Since the production of mathematical knowledge, for example, involves participation and negotiation of meaning within a community of practice, it then means that the use of language as a communicative tool is integral to the process of mathematical enquiry (Siegel & Borasi, 1994). For the socio-cultural view of learning, therefore, language is essential for participation in a community of practice.

¹ The use of this term in this study resonates with the understanding of this term as defined by Kilpatrick, Swafford & Findell (2001)

Language allows meanings to be constantly negotiated and renegotiated by members of a mathematics community – except for the mathematics register which has a fixed meaning across contexts (Brown *et al*; Cole and Engeström, in Chernobilsky *et al*, 2004).

RESEARCH DESIGN AND METHODOLOGY

In order to address the critical question above in this study, a quasi-experimental non-equivalent comparison group design was used because it was not possible to randomly assign learners to groups.

Population and Sample

The study involved a total number of ninety-three learners in grade nine. 45 learners in grade 9A constituted the experimental group while 48 learners in grade 9G constituted the control group (the number 45 and 48 are the number of learners in each respective class in the school). Learners in school offer Sesotho, IsiZulu, IsiXhosa, Setswana and Sepedi and are fluent in one or more of these languages.

The research instrument consisted of 35 questions drawn from a wide range of mathematical content and word problems which learners have covered in the class. They were made up of both multiple-choice questions and questions requiring learners to write the answers. The test items were selected from the 2003 Third International Mathematics and Science Study (TIMSS) and were modified slightly where necessary to suit the context of learners in the study.

Methods of data collection

Data from this study was collected over a period of four weeks. Before the commencement of the ASTRALAB programme in the first week, the mathematics pre-test was administered to both groups. At the end of the implementation phase (in the 4th week), the post-test was administered and the experimental class was videoed. There were class observations of the mathematics class of the experimental group at the beginning of the treatment. The mathematics class of the experimental group was also videoed a week following the end of the treatment. The video-taped mathematics lessons were used to analyse the interaction and communication in the mathematics class.

Implementation Phase of the ASTRALAB Programme

Even though the ASTRALAB programme in itself was designed to be used individually as instructional learning computer software, for implementation in the research school, an adapted version which used an inbuilt projector connected to the computer was used. This enabled whole class instruction and thereby avoided the fundamental criticism of computer based programme instruction as being an individualised approach where instructional situations are cold, mechanical and dehumanizing and where interaction between the teacher and learners is highly eclipsed (Hergenhahn & Oslon, 2001). Learners were required to participate daily in the whole class teaching using the programme for a total of 22.5 hours consisting of

30 sessions in total. In general, the software provides a unique approach to the practice and reinforcement of reading comprehension by building vocabulary, spelling, reading fluency...while testing overall comprehension.

English Pre-test and Post-test

On the first day of implementation, the ASTRALAB programme instructor gave learners in the experimental group a pre-test using the software. He also gave them a post test the week following the end of the implementation of the software. The results showed a 28.2% general improvement in the learners' English proficiency from pre-test to post test.

Nieman (2006) notes that the fact that a learner understands the educator in class and is able to, with ease, read in the language of teaching and learning does not presuppose that such a learner will understand academic texts as easily and write fluently. Earlier research by Cummins (1984, in Nieman, 2006) had led him to draw a distinction between basic interpersonal communicative skills_ (BICS) and cognitive-academic language proficiency_ (CALP). While BICS denotes language proficiency in a social situation and characterised by interpersonal interaction, CALP positions itself as second level of additional language proficiency. This second level of language proficiency is what is needed if learners are to read and understand scientific reports, tasks or academic assignments in general (Nieman, 2006). From an observer point of view, it could be argued that the intervention with the ASTRALAB software, even though very interactive in nature, was biased towards the development of learners' basic interpersonal communicative skills.

ANALYSIS AND FINDINGS

The data collected from the mathematics pre-test and the post-test constituted the quantitative aspect of the study while the data collected from the class observation constituted the qualitative part of the analysis.

Analysis of pre- and post-tests

1. A comparison of the control and experimental groups before the treatment with ASTRALAB ILS indicated that:
 - In the test for skewness, there was an even distribution of learners as far as the mathematics ability is concerned (skewness = .172).
 - The *t*-Test and the nonparametric test indicate that there was no statistically significant difference ($p = .29$) in the test results between the two groups before the treatment (even though there was a difference in the mean scores in favour of the experimental group).

	PRE-TEST RESULTS		POST-TEST RESULTS	
	MEAN	STD DEV	MEAN	STD DEV
EXPERIMENTAL	5.4318	1.9813	6.3556	2.6897
CONTROL	4.9583	2.2874	4.9286	1.9555
p-value	<i>t</i> -Test: .293 nonparametric test: .292		<i>t</i> -Test: .006; nonparametric test: .008	

Table 1: Statistical Results

2. Analysis of the performance in the pre-test and post-test for the control group indicated that there was no statistically significant difference ($p = .84$) between the performances in both pre-test and post-test (even though the performance of learners in this group was lower in the post-test).

3. Analysis of performance in pre-test and post-test scores for experimental group revealed a moderate correlation (.328) between performance in pre-test and performance in the post-test. A *t*-Test and nonparametric test indicated a statistically significant difference ($p = .03$) between scores in pre-test and those of post-test.

4. A comparative analysis of learner performance in both control and experimental groups in post-test indicate that there was a highly significant difference ($p = .008$) between performance in the experimental group compared to performance in the control group.

5. As far as the pre-test analysis by gender was concerned

- There was no statistically significant difference between performance of boys within and across groups. This was also true of the performance by girls in the pre-test.
- In the post-test results for gender, there was no statistically significant difference between performance by boys compared to performance by girls within the two groups.
- There was however a statistically significant difference between the performance of girls in the experimental group and the performance of girls in the control group (there was no difference in the performance of boys across groups in the post-test).

6. As far as the content domains were concerned, there was improvement in all content domains in the experimental group but none of the domains recorded a statistically significant difference.

Analysis of classroom interaction

As noted in previous sections, in addition to algorithmic competence, solving word problems and using mathematical reasoning (Moschkovich, 2002), interaction in the mathematics class is also important in the teaching and learning of mathematics. If the language proficiency of learners was improved, it was also necessary to investigate whether and how such improvement of linguistic competence has either

enabled or constrained the interaction in the class. The coding system for language used by learners and the teacher distinguish when language was used either for questioning, justification, explanation, and regulation. The table below shows the talk distribution between teachers and learners:

	PRE-INTERVENTION		TOTAL	POST-INTERVENTION		TOTAL
	English	African language		English	African language	
TEACHER UTTERANCE	195	10	205	164	5	169
LEARNER UTTERANCE	179	-	179	157	-	157
TOTAL	374	10		321	5	

Note that the above table tends to depict a highly interactive class for both the pre- and the post-intervention lessons. The table tends also to portray that learners talked (almost) as much as the teacher. What the table does not do is indicate the nature or quality of the talk by both the teacher and the learners. A careful study of the pre-intervention lesson and of the post-intervention lesson which takes into consideration the nature of the talk indicates that there was no difference in the interactive pattern of both lessons. What dominated the classroom interaction in both lessons was **first**, only teacher-learner interaction and learner-content interaction. In both lessons, there was no learner-learner interaction. Even though learners were sitting in pairs, the class discussion was not structured in such a way as to encourage learners to share ideas with their partners about their solution process. **Second**, in both lessons, much of the teacher talk was procedural questions requiring the learners to produce short procedural answers.

Discussions and Conclusions from Research

Given that there was a highly significant difference between the post-test scores in the experimental group and those of the control group, and that the experimental group showed a statistically significant higher gains from pre-test to post-test, it can be concluded that the improvement of the performance in mathematics from pre-test to post-test was not due to chance than due to the fact of having improved the English language proficiency of learners. On the other hand, it can be deduced from the data that even though the English language proficiency level of learners was improved, such improvement had no effect on classroom interaction during the mathematics lessons. When one considers these results from the qualitative and quantitative analyses of data, one is tempted to conclude that the two provide conflicting results. If what is foregrounded in the development of language proficiency in learners is the

basic interpersonal communicative skills (BICS) and not both BICS and CALP, there is no doubt that learner interaction in the class – an enterprise which demands that learners debate, reason, critique, analyse, evaluate, express and justify their opinions using academic language in the class – would not be improved. Little wonder that learners did not ask questions in the post-intervention class, and the class did not become less procedural (and more conceptual and adaptive) by way of the nature of talk in the post-intervention lesson. This also means that there is no causal relationship between achievement in mathematics and interaction in the mathematics class. Improvement in performance does not automatically lead to improvement in mathematics communication in class.

What other conclusion can be drawn from the above seeming dichotomy between test scores and interaction in the mathematics class? The present research study is an indication of the fact that proficiency in the language of instruction (English) is closely linked to achievement in mathematics. But improving learner proficiency in English, even though necessary, is not sufficient to impact on classroom interaction. In any classroom, the teacher plays a key role in the management of the interaction in the classroom (Edwards & Westgate, 1987). The teacher's ability, therefore, to draw on learners' linguistic resources - one of which is structuring questions to allow learners to sufficiently express their thinking - is therefore important.

Recommendations

The researcher takes seriously the recommendation by the Centre for Development Enterprise that all mathematics (and Science) activities be “closed linked with improved language [English] education” (CDE, 2004: 33-34). But any attempt to improve the language proficiency of learners with the aim of improving academic proficiency should be done in such a way as to develop concurrently, both the Basic interpersonal communicative skills (BICS) and the cognitive-academic language proficiency (CALP), as Cummins would argue. By so doing, there would be a high possibility of learners' improvement in mathematics learning in English as well as a greater classroom interaction.

Also, appropriate mathematics teacher training (in mathematics) must be accompanied by appropriate training of the teachers in effective English communication (Howie, 2002) and teacher development in strategies of tapping into learners' linguistic resources.

Why was there a statistically significant difference in achievement between boys and girls from the pre-test and post-test results? Was the language proficiency level of girls greatly improved compared to that of boys? What could have been responsible for the difference? Are the comprehension stories, for example, used in the ASTRALAB ILS gender biased? This could be an important area of research for future study as it could provide invaluable information for education software developers.

Limitations of this Study

Akin to the limitation by way of the number of learners involved in the study is the limitation by way of the duration of the implementation phase. Baker (1993) argues that it takes 5 to 7 years to acquire cognitive-academic language proficiency (CALP) in a second language. Therefore, a more prolonged intervention using the ASTRALAB software would have been worthwhile.

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VET IN THE MIDDLE: CATERING FOR MOTIVATIONAL DIFFERENCES IN VOCATIONAL ACCESS COURSES IN NUMERACY

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Within Queensland, young people who disengage from schooling before Year 11, are required to return to study at school or vocational training institutions in special numeracy and literacy access courses if they do not have a job. This paper describes a study of young people's and teachers' perceptions of teaching and learning at one vocational institution. The study found that the young people formed two groups; one which was resistant and resentful, and the other which was participatory and indifferent. The first group had to be cajoled and tempted by intrinsically interesting mathematics activities while the second was happy to work through symbolically-based worksheets. However, regardless of group, most of the access students felt 'cheated' by constructivist approaches using materials; rather they wanted procedurally based activities like the traditional school mathematics classrooms in which they had previously failed.

INTRODUCTION

In 2002 the government of the Australian state of Queensland developed a policy (titled "earn or learn") requiring young people to be either engaged in some sort of work or to be enrolled in further education or training until they are seventeen years old (Queensland Government, 2002). This policy was driven by the need to reduce youth unemployment and to alleviate an ongoing shortage of skilled workers. Vocational training institutions called Technical and Further Education [TAFE] institutions in Australia had been providing numeracy access courses for these disengaged young people whose achievement levels were too low to meet numeracy prerequisites of traineeships and apprenticeships. Australian young people who leave school early generally have low numeracy skills and form a major group within these courses and the unemployed (Millar & Kilpatrick, 2005).

Numeracy skills, disengagement and re-engagement

Although there is strong support for the importance of numeracy in training and employment (Department of Education, 2004; Fitzsimons, 2001; Karmel, 2005), there is little research that provides insight into what numeracy skills are required for employment and how they can be effectively taught (McLeish, 2002). There is some evidence that flexibility with clients' needs, contextualising numeracy to the culture and background of young people (Fitzsimons, 2002; Millar & Kilpatrick, 2005), the use of non-scholastic, kinaesthetic, individualised activities to link numeracy with vocational interests and illuminating the importance of numeracy in a holistic way (McNeil & Smith, 2004) may be effective approaches. However, as McNeil and Smith (2004) argue although low-achieving, disengaged young adults are unique in their learning needs, and need to be hooked or lured into attempting educational tasks, the contextualisation (which is the basis of the hooks and lures) has to be balanced with the

learning priorities of the classroom. This balance is particularly difficult when young people are in TAFE. First, TAFE colleges use competency-based training frameworks which tend to compartmentalise or atomise numeracy topics. Second, as Boaler (1993, cited in Fitzsimons, 2002, p. 148) argues:

… random insertion of contexts into assessment questions and classroom examples in an attempt to reflect real-life demand and to make mathematics more motivating and interesting … ignores the complexity, range and degree of students experience's as well as the intricate relationships between an individual's previous experience, mathematical goals and beliefs.

According to the Australian National Training Authority [ANTA] (2004, p. 27) “‘tick and flick’ training, is more than a rare occurrence in TAFE classrooms” rather than teaching of higher-order thinking skills with high-quality learning outcomes.

Research indicates that young people disengage from school in Australia because of school practices (e.g., uninspiring pedagogy and teaching, unfair treatment and disrespect from teachers, and inconsistent discipline - (Smyth, et al., 2000) and the economic and academic vulnerability of their low socio-economic status (Finn & Rock, 1997). Low socio-economic students do not have access to the communicative strategies that middle class students have such as Lemke's (1990, cited in Zevenbergen, 2000) triadic dialogue which involves teacher question; student response and teacher evaluation of students' response and works to manage student behaviour and stipulate class content. This type of dialogue is never explicitly taught to students and exposure to the same dialogue at home makes it easier to learn, however this is rare for students from lower socio-economic backgrounds (Zevenbergen, 2000).

Re-engagement of young people requires the interplay of participation and reification giving shape to their mathematical understandings. This, according to Ewing (2005) is assisted by teaching practices that take account of students' different learning styles. For example, linking numeracy to real life, and ensuring that learning is paced so that concepts are understood before further ones considered.

Methodology

The study described in this paper is part of a larger Australian Research Council grant. The project was funded to investigate the mathematics teaching and learning of low-achieving post-Year Ten students in order to develop theories regarding effective materials for teaching mathematics in access courses within TAFE's, secondary schools. The project explores the effectiveness of utilising everyday and vocational contexts to teach the basic mathematics needed by these students. The methodology used in the project is primarily qualitative, interpretative and intervening (Burns, 2000). A case-study approach (Yin, 1989) is used to investigate what happens when researchers, TAFE, and school teachers collaborate to improve the teaching and learning of mathematics.

The participants for the study consisted of thirty-five students, two teachers and two tutors from a TAFE numeracy access course. Many students attended the course under

directions from the Queensland Magistrates courts; a significant number had been expelled from previous schools or had ‘dropped out’ due to confrontation with traditional schooling environments. Students could be considered part of a “street kid” sub-culture. The access numeracy course had been developed in recognition of the importance of having tailor-made and flexible choices for young people disengaged from high school. Its basic premise was to increase fundamental skills, revive engagement in training and learning and to provide some routine to prepare students for workplace entry or further study.

Data was gathered through observations of access numeracy classes at the TAFE and semi-structured interviews with students and teachers. The foci of the interviews were students’ and teachers’ prior numeracy learning and teaching experiences and their perceptions of numeracy, the learning and teaching of numeracy, and the access numeracy course.

At the commencement of the project the classes were videotaped for later analysis. Interviews with teachers and students were then conducted. All the recorded data was transcribed and analysed for commonalities. Categories were developed through an evolving process of refinement. These were classified in preparation for intervention. The findings from these data are presented.

Results

Approximately thirty-five students were enrolled in the numeracy access course in two classes with teachers Mary and Kyle who were assisted by one or two tutors.¹ The students were predominantly fifteen to sixteen years old with equal proportion of male and female students and a few Indigenous students. Most of the thirty-five students had difficulties within traditional learning environments, low numeracy and literacy skills (about mid primary or elementary level), and irregular attendance. Social and significant personal problems were also identified. Most students were also enrolled in courses in literacy, computers and vocational skills.

Observations showed that, although student behaviour was, in general not positive for learning, there were two categories of students. The first category consisted of students whose behaviour was resistant and resentful or, in Ewing’s (2005) terms, non-participatory. These students took time to settle into class, and had fluid and inconsistent engagement in learning. They actively refused to follow teachers’ instructions and argued about doing work (asking why they were doing the work and how it was relevant to them), refused to do anything they felt was “childish”, copied other students’ work, and behaved dangerously with materials (e.g., throwing scissors, slapping rulers), preventing hands-on activities. They needed constant reminders from a tutor to stay focused, gave up easily when they encountered difficulty and disengaged from the task after a short time span. They openly told peers and staff that “maths sucks”, that they “sucked at maths” and hated mathematics. They had a negative

¹ All names used in this paper are pseudonyms.

attitude to mathematics which is reflected in the words of one student – “I can’t do this! I don’t want to! This is too hard!” However, they were less antipathetic towards one-to-one tutoring, self-paced workbooks, and peer and teacher interactions that were supportive, respectful and equal. Finally, their perceptions of numeracy were negative as this interview between the researcher (R) and Jess showed.

R: Do you feel like maths is important for you in the future?

Jess: Yes, ‘cos (because) I need to use measurement and volume in plumbing. … I don’t enjoy maths at all.

R: What about when you get the right answer?

Jess: It doesn’t bother me. I just don’t like it.

R: Have you ever thought of yourself as someone who’s good at maths?

Jess: Nope, never.

R: Is there anything you don’t like about the maths here in the TAFE class?

Jess: Just doing stuff that I don’t need to know.

R: What sort of stuff is that?

Jess: I can’t remember … but the stuff that we’ve been doing in the past week. … It was just annoying ‘cos (because) I already knew it. I don’t see why I had to write it down again.

The second category consisted of students whose behaviour was participatory yet indifferent. In general, these participatory students were more likely to ask for help and to use the tutors productively. They engaged in tasks more easily and remained focused for longer periods and did not give up as quickly as the resistant learners. They worked through set booklets at their own, and were willing to tutor peers. They showed little interest in numeracy *per se* or the numeracy activities, but, put up with the required workbooks in order to complete the course. Like the resistant students, they also liked one-to-one tutoring and supportive environments. Their perceptions were more positive, but still showed an underlying dislike of mathematics as the following two interview excerpts between the researcher (R) and Sam and Taylor showed.

Sam: Well, I don’t like maths but it has to be done because you use it all through your life.

R: Do you think it’s important?

Sam: Yeah, it’s important … ‘Cos (because) I need it so I can go through life knowing what to do instead of going to the shop and getting confused and not knowing how much it is.

R: If you hate maths, why do you come along every week?

Taylor: Because I want to get my certificate so I can get a good job.

R: And you think that the maths …

Taylor: It will help me in the future … But I hate maths but this is alright. I’ve always hated maths but this is the best maths that I’ve had.

Observations showed that the predominant teaching practices evident included worksheets, workbooks, and project work. Learning spaces were basic and

uninteresting with minimal support material and lack of equipment. The teachers had difficulty maintaining general engagement and directing the class because of individual students' demands. They rarely took disciplinary action (and when they did, they found it difficult to implement and sustain). Both teachers and tutors had little training in how to teach numeracy at the level (middle primary) of the students. They tended to focus more on achieving a friendly environment than challenging the students mathematically.

The observational notes showed that one teacher approached the resistant and participatory students differently. The first teacher observed, Mary, tried hard to build positive relationships with the students, Kym and Dani, as these excerpts from observations showed.

A number of resistant students arrived late, talking loudly over the teacher, and throwing things around the room and swearing. Mary did not respond to these behaviours. She attempted to continue the lesson, trying to engage the students by using soft words. She called one by name, "Kym" and asked "Do you have a worksheet? What are you doing?" The teacher handed out a worksheet, to which Kym said, "I hate it ... Why do they make us do this? I don't know this!" The teacher was unable to respond as she has been distracted by another student (but a tutor manages to get the student on task). Two girls requested to leave early. The teacher reminded them that they left early last week and that this should be the last time. One girl interrupted her, "We'll be here other times." The teacher replied, "Do you promise?" The girls answered with a mildly interested "Yeah" and walked away. Dani appeared disinterested in the work. Mary attempted to engage him, saying, "How about you grab one of those sheets ... how about you come over here and sit down?" He took a seat at a table with the teacher and four other students. However, when the teacher's attention was focused on other students, Dani got bored, stood up and proceeded to walk around the classroom. At this point, three other students left early without an explanation. The teacher was unable to stop them.

Mary's approach to the participatory students, Eli and Frank in the same lesson, as shown in the excerpt below, was different to the resistant students in terms of the attention that she gave to the students.

Eli and Frank were working quietly and apparently efficiently; they did not appear distracted by all the commotion in the classroom. Once the disruptive students had left, they were quick to take advantage of the teacher's newly available attention and asked many questions. The teacher did not attend to these students until the class had quietened down.

The second teacher, Kyle was more traditional; he focused on numeracy content in a procedural manner and attempted to push on with the lesson regardless. He did this for both the resistant student, Andy, and the participatory student, Ben. Kyle appeared to have only one explanation for given problems as these three interchanges between the teacher and the students showed.

Kyle: Can I show you the next step with the ruler? See how you've got the top one right?
Let me show you the next ones, so when you're adding together the nine and two ... Andy? Andy? So we want to add together nine ...

Andy: No I don't want to do it!

Kyle: No come one, just watch for a minute, that's all ... come on, there's nine plus two that makes eleven. Now you can't fit eleven underneath that, like you did before, so you put down the one, off the eleven and you carry one to there, so that's the eleven there now, split up into one and one, now you need to add together one, six and seven ...

Andy: I feel like I'm in school again ...

Kyle: Pardon? Andy: I feel like I'm in school again.

Kyle: Oh well not for long, because I'm only going to show you once and then you'll be able to do it on your own ...

[The teacher returns later to check on Andy's progress.]

Kyle: Okay so you know how to add up now?

Andy: No, I just copied it off him ...

Kyle: Why'd you do that?

Andy: Because I hate maths.

Kyle: Okay but you could do it with the ruler, couldn't you?

Andy: (Yawns really loudly!)

[Ben asks for clarification in a division task.]

Kyle: So you go down till you get to the number nearest to 26, which is 24. So when it says 3 lots of 8 are 24, it goes in 8 times with a remainder of two.

Ben: Okay ... I'll try and do this one.

When Ben still struggled to understand, Kyle explained the procedure again but in the same way as the first explanation. The tutor, Jasmine, was then left to address the student's difficulties as this excerpt of an observation showed.

Kyle again worked through the problem procedurally for the student using almost identical language and instruction. Kyle left Ben to Jasmine who was able to elicit responses from the student and encouraged him to come to his own conclusions. They solved the problem and then tried to identify where difficulties arose for the student. Jasmine eventually explicitly told Ben that trying to understand the problem is too hard and that he should only worry about ROTE learning the procedure. She said to do it "in your head, rather than trying to understand why, because that's impossible, just try to remember that if you put a naught above that number ..."

There was inconsistency between the teachers and tutor instruction. It appears that some teachers and tutors were not trying to develop understanding and meaning in the mathematics.

DISCUSSION AND CONCLUSIONS

The conclusions from the study indicate there are two distinct categories of student and that neither is having their numeracy needs satisfactorily met within the TAFE. Both categories of students disliked numeracy but the participatory students were willing to put up with the numeracy lessons if it enabled them to undertake vocational training. The resistant students were not so willing, and had behavioural problems - disrupting

the class and attracting the teachers' attention. In this situation, and despite Mary's intentions, she was not successful with her teaching because of the competing demands of the students. The resistant students did not want to study in any situation. Rather, they continued to disrupt the class. The participatory students' needs were not met either because of the disruption and included requests for regular sequenced worksheets. Consequently any attempt at implementing mathematical investigations with hands-on material was thwarted. The repetitive procedural approach used by Kyle was also not successful as the participatory students appeared to need more variety with explanations. Individual attention appeared to work for most students but the TAFE institution did not have the resources (often they were only able to provide one teacher and one tutor per class of twenty).

In the TAFE context the teachers were in the middle of competing forces; what seemed to work for one of the two categories was ineffective for the other. This situation was exacerbated by the lack of resources, the uninspiring learning spaces, familiarity and success in traditional numeracy teaching interactions, and the competing need to make progress through outcomes. The TAFE numeracy access course reflected the findings of Fitzsimons (2002), and McNeil and Smith (2004).

Most interesting of all, the students' perceptions of numeracy and the TAFE classrooms indicated that they still believed in the traditional absolutist and procedural ideas about numeracy and its teaching that had resulted in their previous failures (and that their beliefs appeared to be much stronger than those of high achieving students). As well, the behaviour of the resistant students appeared to be a result of rejection of education's capacity to provide social mobility and acceptance of an unskilled labouring or unemployed role in society within which they could enact local control, a "learning to labour" sub-culture as in Willis (1978).

The social conditions and resources at the TAFE make any intervention problematic. However, the "earn or learn" context provides a framework for emancipation as well as oppression. First, the students have to come to the institution and be given opportunities for engagement. Second, the interests of the students is located within street life and vocational improvement enabling a "street maths" vs "school maths" or "vocational maths" vs "school maths" approach to have some resonance.

To take advantage of this, it is evident that the two types of students need to be separated into different classes. The participatory students need to have their numeracy horizons widened from workbooks to concepts and strategies. One possibility for this process is through a focus on the resilience that has brought them to this position in relation to the vocation they wish to take up. Questions such as, what gives these students the strength to rebuild their lives, where is the numeracy that underpinned this resilience, what numeracy do they need for their vocation, and how can this be linked to formal pre-vocational mathematics?

The resistant students need to look inward but in a different manner. They need to develop an identity (and pride) that allows them to see themselves as controlling their

wider world. Numeracy focusing on the social situations that control them (e.g., money, employment, and police) could be a starting point.

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THE GENDERING OF MATHEMATICS IN ISRAEL AND AUSTRALIA

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The “Mathematics as a gendered domain” instrument was administered to grade 9 Israeli Jewish and Arab students. The data were examined for differences in the views of the students from the two ethnic groups and also among males and females within each group. The instrument was designed and normed in Australia and the Israeli data were compared to the Australian findings. Differences between the cultural groups in Israel were identified; the Jewish students’ views were more similar to the Australians’ than were the Arab students’. The implications of the findings are discussed.

BACKGROUND

Gender inequities in mathematics education are frequently reported with respect to achievement, participation rates, and in regard to students’ attitudes and beliefs about mathematics (e.g., Leder, Forgasz & Solar, 1996). While some changes have been reported over time in stereotyped attitudes and beliefs (e.g., Forgasz, Leder & Kloosterman, 2004), PISA 2003 results (OECD, nd) revealed that males generally outperformed females, while gender differences were found for some countries but not in others in TIMSS 2003 (Mullis, Martin, Gonzalez, & Chrostowski, 2004). Participation data in Australia (Forgasz, 2006) still reveal that males outnumber females in the most challenging mathematics subjects offered at the grade 12 level.

In Israel, the educational systems for Arabs and Jews are segregated, although both are run by the Ministry of Education (Birrenbaum & Fasser, 2006). Hebrew and Arabic are the languages of instruction in the pertinent systems, while the intended mathematics curricula are the same. Ayalon (2002) maintained that Jewish students generally had greater choice of advanced level subjects leading to matriculation than students in Arab schools, where advanced courses were often limited to mathematics, sciences, and history. The restricted curriculum in Arab schools was hypothesised to benefit females with respect to access to *valued* knowledge such as mathematics (Ayalon, 2002). Mittelberg and Lev-Ari (1999) reported that Arab girls’ preparedness “to adopt a mathematically-based profession in the future is particularly high both when compared to Jewish girls as well as Jewish boys” (p.88).

Based on data from the Israeli Central Bureau of Statistics, it was found that “gender inequality among Arab students was relatively moderate, with higher proportions of Israeli Arab than Jewish girls taking advanced courses in mathematics (Ayalon, 2002). Ayalon (2002) cited findings indicating that between 1948 and 1980 Arab females’ enrolments at all levels of secondary education had equalled males’ and that they had higher participation rates in post-secondary education. Mittelberg and Lev-Ari (1999) reported that Arab females also had high levels of perceived achievement and

self-confidence in mathematics and were willing to consider mathematically-based studies and professions in the future. However, cultural factors appear to work against Arab females being able to capitalise on their potential. Compared to the Jewish population in Israel, the Arab population is more conservative and “Arab women are not expected to be active outside their homes and labour market participation is still low” (Ayalon, 2002, p.63). Indeed, while females comprise 56% of all Arab matriculants, only 20% of all Arab women are found in the labour force, compared to 51% of all Jewish women and 65% of all Arab men (Fogel-Bizau, 2003).

Jewish students in Israel appear to follow more closely the gender-stereotyped patterns of participation reported widely in other western nations such as the USA and Australia. That is, more males than females are enrolled in the most challenging mathematics courses offered at the secondary level (Mittelberg & Lev-Ari, 1999). In 2003, Australian and Israeli grade 8 students participated in TIMSS. The Australians (ranked 14) performed slightly better than the Israelis (ranked 19) of the 46 countries participating. In both Israel and Australia, there were no significant gender differences in performance, although males did slightly better than females (see Mullis et al, 2004). It must be recognised that Israeli Arabs comprise only 20% of the Israeli population and thus the Israeli TIMSS results reflect more closely the Israeli Jewish students’ performance; there is no way of disaggregating the TIMSS data by ethnic groups to examine the performances of Israeli Jews and Israeli Arabs.

In the study reported here Israeli Jewish and Israeli Arab grade 9 students’ beliefs about the gender stereotyping of mathematics were compared. Within group gender comparisons were also undertaken. Data were also gathered on perceptions of mathematics achievement levels which were compared by ethnic group and by gender within ethnic group. Further comparisons were made with Australian grade 9 student data gathered in an earlier study (see Forgasz, Leder & Kloosterman, 2004).

AIMS AND METHODS

The *Mathematics as a Gendered Domain* instrument was developed by Leder and Forgasz (see Leder & Forgasz, 2002 for details) and normed on Australian students in grades 7-10 that included 253 (122 males, 131 females) grade 9 students. It is comprised of three subscales with 16 Likert-type items scored on five-point scales from SD=1 to SA=5 on each. The three subscales are: *Mathematics as a Male Domain* [MD], *Mathematics as a Female* [FD], and *Mathematics as a Neutral Domain* [ND]. Sample items from each scale include:

MD: Boys understand mathematics better than girls do

FD: Girls are more suited than boys to a career in a mathematically-related area

ND: Boys are just as likely as girls to help friends with their mathematics

The instrument was translated into Hebrew and into Arabic and administered to 103 grade 9 students in 4 Jewish schools and 112 grade 9 students in 3 Arab schools in Northern Israel. While recognising that this purposeful sample was not representative

of the entire Israeli grade 9 population, it enabled robust comparisons to be made between the beliefs of Israeli Jewish and Arab students, and was large enough to make comparisons with the earlier gathered Australian data. Also included on the instrument was an item asking students “How good are you at mathematics” [HGM]. Students responded on a five-point scale ranging from 1=weak to 5=excellent.

RESULTS

Israeli and Australian grade 9 students' perception of mathematics achievement

Mean scores for perceived levels of mathematics achievement [HGM] were compared by country using an independent groups t-test. The Israelis were found to have a significantly higher mean level of perceived mathematics achievement than the Australians (Israelis: 3.67, Australians: 3.43, $p<.01$). These findings are inconsistent with the TIMSS findings that revealed that the Australians had performed better than the Israelis. As noted earlier, Israeli Arabs are a minority of the Israeli population and the difference is most likely due to the unrepresentative sampling of Israeli students, that is, using approximately equal numbers of Jewish and Arab Israelis. The Israeli data were disaggregated by ethnic group, and comparisons between the three ethnic groups were undertaken.

Perceptions of mathematics achievement: Ethnic comparisons

A one-way ANOVA was conducted to determine if there were differences in mean scores on students' perceptions of their mathematics achievement levels by ethnic grouping: Israeli Jews, Israeli Arabs, and Australians. Scheffe post-hoc tests were used to determine which pairs of mean scores were significantly different. The results are shown in Table 1.

Ethnicity	N	Mean	F, p-level	Scheffe post-hoc
Israeli Jews	103	3.59	4.15, $p<.05$	Israeli Jews – Israeli Arabs: ns
Israeli Arabs	107	3.75		Israeli Jews-Australians: ns
Australians	250	3.43		Arab-Australians: $p<.05$

Table 1. HGM: ANOVA results by ethnic group

As can be seen in Table 1, there was a statistically significant difference between the three groups ($p<.05$). The post-hoc tests revealed that this difference was due to the difference between the mean scores for Israeli Arabs and Australians ($p<.05$) with the Israeli Arabs believing they were higher achievers (mean=3.75) than Australians (mean=3.43). There were no significant differences in mean scores for the Israeli Jews and Israeli Arabs or for the Israeli Jews and Australians.

Gender differences within each ethnic group were examined using independent groups t-tests. The results are shown in Table 2 and show that the only gender difference was found among Australian students with males believing they were higher achievers than females (M mean=3.56, F mean=3.30).

Table 2. HGM: Means and t-test results by gender within ethnic group

Ethnicity	Male		Female		T, p-level
	N	Mean	N	Mean	
Israeli Jews	50	3.56	52	3.60	ns ¹
Israeli Arabs	57	3.81	54	3.67	ns
Australians	121	3.56	129	3.30	-2.28, p<.05

¹ ns = not significant

In summary, the Israeli Arabs had the highest perceived mathematics achievement levels among the three groups. The beliefs of the Israeli Jews about their mathematics achievement levels were similar to those of the Australians, consistent with the TIMSS 2003 data. Unlike the Australians, however, there were no gender differences in the beliefs of Israeli Jews with respect to mathematics achievement levels.

Mathematics as a gendered domain: Differences by country

The grade 9 students' scores on the MD, FD and ND subscales were compared by country using independent groups t-tests. Statistically significant differences by country for mean scores on each of the three subscales were found. The mean scores are illustrated in Figure 1.

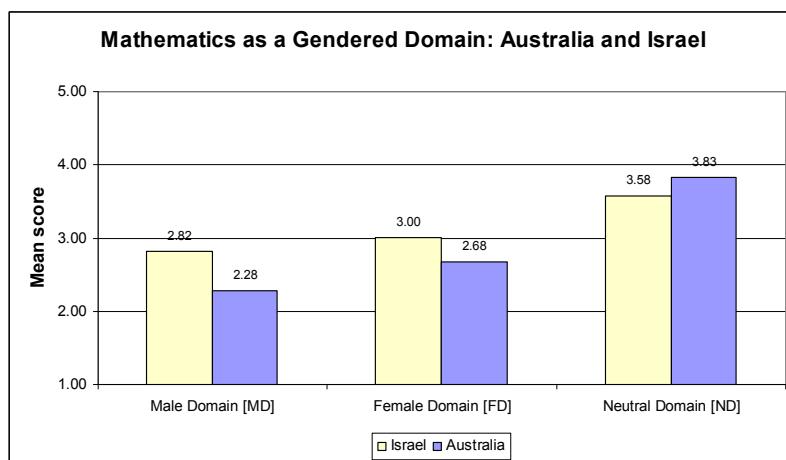


Figure 1: MD, FD and ND: Mean scores for Australian and Israeli students

The data in Figure 1 indicate that the Israeli students:

- Disagrees less strongly that the Australian students that mathematics was a male domain (Australia: mean = 2.28; Israel mean = 2.82)
- Were unsure if mathematics was a female domain while the Australians did not believe that it was (Australia: mean = 2.68; Israel: mean = 3.00)
- Agreed less strongly than the Australians that mathematics was a neutral domain (Australia: mean = 3.83; Israel: mean = 3.56).

Gendered beliefs about mathematics: Ethnic differences

The mean scores for each ethnic group on each of the three subscales of the *Mathematics as a gendered domain* scale are shown in Figure 2.

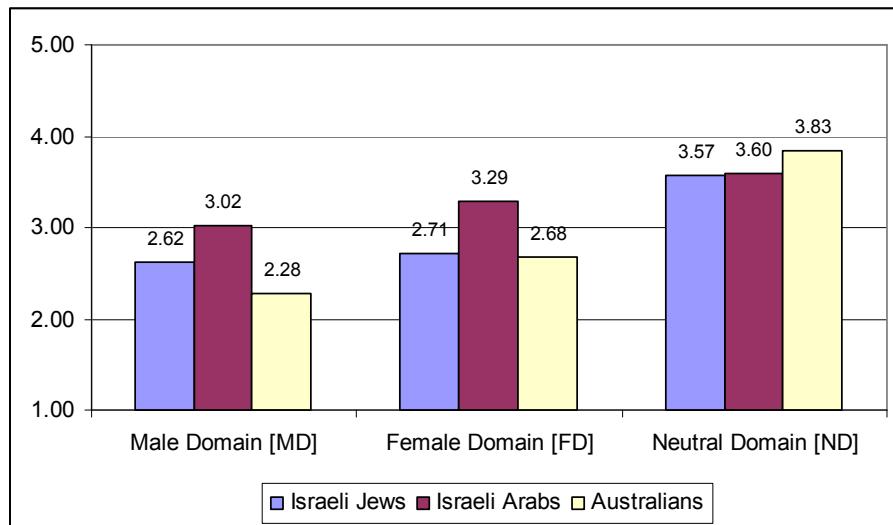


Figure 2. MD, FD, and ND: Mean scores by ethnic group

As can be seen in Figure 2, the order of the belief measures on the three subscales was the same for all three groups: highest score on ND, and lowest score on MD, that is, they agreed most strongly that mathematics was a neutral domain and least strongly that mathematics was a male domain. While all three groups strongly agreed that mathematics was a neutral domain (means $>> 3$), for the MD and FD subscales, the directions of the beliefs of the Israeli Arabs differed from those of the Israeli Jews and the Australians. The Israeli Arabs were unsure if mathematics was a female domain (mean approx. 3) whereas the other two groups disagreed that it was (means < 3), and the Israeli Arabs believed that mathematics was a male domain (mean > 3) whereas the other two groups disagreed that it was (means < 3).

ANOVAs followed by post-hoc tests were conducted to determine if there were statistically significant ethnic differences in the students' gendered beliefs on the three *Mathematics as a gendered domain* subscales. The results are shown in Table 3. The data revealed that there were statistically significant differences in the mean scores on the MD, FD, and ND subscales, and post-hoc tests indicated that the only non-significant differences in mean scores were for Israeli Jews and Australians on the FD, and for Israeli Jews and Israeli Arabs on the ND.

The statistically significant findings indicated that:

- MD: Australians (mean = 2.28) disagreed more strongly than Israeli Jews (mean = 2.62) that mathematics was a male domain; Israeli Arabs were not sure if mathematics was a male domain (mean = 3.02)
- FD: Australian (mean = 2.68) and Israeli Jews (mean = 2.71) disagreed that mathematics was a female domain; Israeli Arabs agreed that mathematics was a female domain (mean = 3.29)
- ND: All three groups agreed that mathematics was a neutral domain (means >3), with the Australians (mean = 3.83) agreeing more strongly than Israeli Jews (mean = 3.57) and Israeli Arabs (mean = 3.60).

Subscale	Ethnicity	N	Mean	F, p-level	Scheffe post-hoc
MD	Israeli Jews	103	2.62	42.26, <.001	Israeli Jews-Israeli Arabs: p<.001
	Israeli Arabs	107	3.02 ¹		Israeli Jews-Australians: p<.001
	Australians	233	2.28		Israeli Arabs- Australians: p<.001
FD	Israeli Jews	99	2.71	36.42, <.001	Israeli Jews-Israeli Arabs: p<.001
	Israeli Arabs	100	3.29		Israeli Jews-Australians: ns
	Australians	236	2.68		Israeli Arabs- Australians: p<.001
ND	Israeli Jews	101	3.57	13.89, <.001	Israeli Jews-Israeli Arabs: ns
	Israeli Arabs	104	3.60		Israeli Jews-Australians: p<.001
	Australians	224	3.83		Israeli Arabs- Australians: p<.001

¹ One sample t-test indicated that this mean was not significantly different from 3

Table 3. MD, FD, and ND: Means and ANOVA results

Gendered beliefs about mathematics: Gender differences

To determine if there were gender differences in the grade 9 students' gendered beliefs about mathematics within each ethnic group, independent groups t-tests were conducted. The results are shown in Table 4.

Ethnicity	Subscale	All	Male		Female		t, p-level
			N	Mean	N	Mean	
Israeli Jews	MD	2.62	49	2.51	53	2.73	ns
	FD	2.71	50	2.59	48	2.86	2.346, <.05
	ND	3.57	49	3.57	51	3.56	ns
Israeli Arabs	MD	3.02	55	2.81	51	3.25	3.241, <.01
	FD	3.29	49	3.39	51	3.20	ns
	ND	3.60	49	3.58	54	3.62	ns
Australians	MD	2.28	112	2.47	121	2.11	-4.35, <.001
	FD	2.68	115	2.66	121	2.70	ns
	ND	3.83	108	3.79	116	3.87	

Table 4. MD, FD, and ND: Means and t-test results by gender within ethnic group

A few interesting trends are apparent from the data in Table 4.

- There was a clear similarity in the belief patterns of the Australian males, Australian females, Israeli Jewish males, and Israeli Jewish females. All held that mathematics was a neutral domain (means > 3) and disagreed that it was either a male or a female domain (means < 3).
- The pattern was different among the Israeli Arabs. The females believed that mathematics was a neutral domain as well as being both a male domain and a female domain (means > 3), and the males agreed that mathematics was a neutral and a female domain (means > 3), but disagreed that it was a male domain (mean < 3).

The three statistically significant gender differences, one for each ethnic group, showed that:

- Israeli Jews: Males (mean = 2.59) disagreed more strongly than females (mean = 2.86) that mathematics was a female domain [FD]
- Israeli Arabs: Females agreed (mean = 3.25) and males disagreed (mean = 2.81) that mathematics was a male domain [MD]
- Australians: Females disagreed more strongly (mean = 2.11) than males (mean = 2.47) that mathematics was a male domain [MD]

In summary, the Israeli Jews and the Australians (as well as the males and females in each country) held similar views. They agreed that mathematics was a neutral domain and rejected mathematics as either a male or a female domain. The findings for the Israeli Arabs' were quite different and are ambiguous. They agreed that mathematics was both a neutral and a female domain, and were uncertain if it was also a male domain (mean = 3.02). The gender difference on the MD subscale appears to explain the overall uncertainty among the Israeli Arabs; while the females agreed that it was a male domain (mean = 3.25), the males disagreed (mean = 2.81).

CONCLUSIONS

Consistent with the TIMSS findings for mathematics performance (Mullis et al., 2004), the beliefs about mathematics performance levels and the gendered beliefs about mathematics of all the Israeli students and the Israeli Jewish students but not the Israeli Arab students were similar to the Australian students'. On average, the Israeli Jews and the Australians believed they were above average in mathematical performance, believed that mathematics was a neutral domain, and disagreed that it was either a male or a female domain. These findings suggest that there is a cultural similarity between the Israeli Jewish and the Australian grade 9 students. The statistically significant differences between the mean scores of the two groups indicated that the Australians' beliefs were more strongly held. That Australian society may be less gender stereotyped than Israeli society may partly explain these findings but further research is needed. It is noteworthy that the Israeli Jewish students' views were consistent with findings from the *Mathematics as a gendered domain* scale in the USA as well as Australia (Forgasz, Leder & Kloosterman, 2004).

Of the three groups, the Israeli Arabs had the highest perceived levels of mathematics performance, a finding that may be partially explained by the limited curriculum offerings in Israeli Arab schools (Ayalon, 2002) that may limit students' perceptions of which discipline areas they are relatively stronger or weaker. The Arab Israeli students' gendered views of mathematics were intriguingly different from the views of the Israeli Jews and the Australians and are not easily explained. Like the others, the Israeli Arabs (both males and females) also believed that mathematics was a neutral domain. However, both the males and females simultaneously believed that mathematics was a female domain and, while the Israeli Arab males disagreed, the Israeli Arab females agreed that mathematics was a female domain. These apparently

ambiguous findings strongly suggest that there may be a cultural factor within the Israeli Arab community in which there conflicting beliefs associated with the gender stereotypes of the adult community and what takes place in classrooms and school settings. There may be an unresolved tension between school and post-school expectations and employment options for both male and female Israeli Arab students. Clearly more research is needed to understand better the ambiguity in the Israeli Arab students' findings about the gendering of mathematics.

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21ST CENTURY CHILDREN, NUMERACY AND TECHNOLOGY: AN ANALYSIS OF PEER-REVIEWED LITERATURE

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Technology has catapulted young children into a society where numeracy practices are integral to their everyday lives and future success. In order to determine the scope and foci of the literature on early childhood digital-numeracies, this study examines the articles published between 2000 and 2005 in the ERIC database and proceedings of the annual conferences of the International Group for the Psychology of Mathematics Education [PME]. Overall, this study revealed (1) a lack of peer-reviewed articles that discuss, investigate, or examine early childhood digital-numeracies; (2) an absence of studies on the prior-to school years, and (3) an absence of research exploring the impact of new technologies on young children's numeracy practices.

INTRODUCTION

Young children are being born into a world that is built on digital technology — a world where having competence and the disposition to use mathematics in context is essential. Considering the widespread demand for a numerate citizenry in a digital age, it is essential that young children develop the foundations of digital-numeracies. Throughout this paper, the term, “digital-numeracies” is used as a parallel term to digital-literacies (Lankshear et al., 1997). Thus, “digital-numeracies” are the numeracy practices, behaviours and events which are mediated by new technologies. These technologies comprise technological innovations that have been made possible through digitization, such as digital music players. They also include “old” technologies (e.g., digital television) which have been transformed through the digital signal (Marsh et al., 2005). Thus, research on children’s numeracy learning and digital technologies is particularly timely.

Research findings on children’s early mathematical growth (e.g., Baroody, Lai, & Mix, in press) together with the growing number of children who spend time in early childhood programs has created an impetus for the creation of policies, curricula and guidelines that support the development of early years care and education (Organisation for Economic Co-operation and Development [OECD] 2006) including mathematical proficiency (Ball, 2004; Clements, Sarama, & Di Biase, 2003; National Association for the Education of Young Children [NAEYC], 2002; National Council for Teachers of Mathematics [NCTM], 2000). Increasingly, digital technologies are impacting on everyday life; hence, children’s mathematical proficiency needs to be considered in relation to their learning with and from these technologies. Thus, the purpose of this paper is to examine the current status of peer-reviewed literature pertaining to young children’s (birth to eight years) digital-numeracy practices and engagement with new technologies. The outcomes of this review will establish the

scope and adequacy of the literature base and provide directions for future research on digital-numeracies in the early childhood years.

21ST CENTURY CHILDREN, NUMERACY AND TECHNOLOGY

Converging Trends

The past decade has seen the emergence of three understandings related to young children's mathematical learning in the 21st century. First, the early years of life have been highlighted as fundamental to lifelong learning and it has been acknowledged that long-term success in learning and development requires quality experiences during the "early years of promise" (Carnegie Corporation, 1998). Second, the NCTM (2000) has advocated the salient and powerful nature of mathematical proficiency stating that "mathematical competency opens doors to productive futures – a lack of mathematical competencies keeps those doors closed" (p. 5). Contemporary views and theories acclaim that young children are capable of mathematical competencies that are extensive and impressive (Clements & Sarama, *in press*). Third, the influence of technology on human life in the new millennium has created a world characterized by diverse and energetic communication, vast amounts of information, and rapid change. Technology affects the daily lives of every person, directly or indirectly (Williams, 2002). The coalescence of these three understandings (i.e., early learning, mathematical proficiency, technology) indicates a need to consider what we know about the development of young children's digital numeracies. As a prelude to the examination of the literature young children's digital numeracy practices, a brief overview of 21st children and the relationship between numeracy and technology is presented.

21st Century Children and their Learning

Most children of the western world have access to technology and devices that impact on their lives – whether they be for entertainment (eg., Playstation©) or day-to-day living (e.g., microwave) or in school (e.g., computers). The range of these technological devices is expanding and includes console games, digital music players, video cams, mobile phones, and various digital toys. As the roots of later competence are established long before school age (Bowman, 2001) and findings from neuroscience confirm the importance of the connection between young children's experiences and achievements later in life (Bruer, 1999), it is important to consider how learning occurs from birth to eight in a variety of contexts in addition to school. Opportunities for mathematical experiences and interactions with technology occur before children begin school and in parallel with schooling. Thus, research on the development of digital-numeracies in young children needs to span various ages and learning contexts.

Numeracy and Technology

The skills, knowledge, and abilities needed to participate and succeed in 21st century society are vastly different to those needed in the previous century. The amplified need

for numeracy is a result of the demands of the technologically-oriented age (Her Majesty's Inspectorate, 1998; NCTM, 2000). For example, Steen (1997) argues that modern life is dominated by technology, digital tools and devices, and that "Numeracy is the currency of modern life" (p. xvii). Steen (2001) also credits the rise in quantitative data, numbers, and information to the universal increase in the usage of technology, computers, and the internet. Civil rights leader Robert Moses argues that mathematics has become a humanitarian issue stating, "children who are not quantitatively literate may be doomed to second class economic status in our increasingly technological society" (cited in Schoenfeld, 2002, p. 13). Clearly, numeracy is no longer the fortune of the elite but a requirement for all citizens. Similarly, Malcom (1999) subscribes to the opinion that mathematical achievement in a technological and global society will have a major impact on students' career aspirations, their role in society, and even their sense of personal fulfilment. Thus, the need to understand and to use mathematics and technology is fundamental to 21st century life.

METHOD

This paper reports on the status of peer-reviewed literature that focuses on early childhood mathematics and technology through an investigation of contemporary literature in the Education Resources Information Center [ERIC] database and the annual conference proceedings of the International Group for the Psychology of Mathematics Education [PME]. ERIC was selected as widely accessible general database that provides free access to more than 1.2 million bibliographic records of educational journal articles and other education-related materials. The PME conference proceedings were selected because PME specialises in the exchange, promotion, and stimulation of scientific information and interdisciplinary research in the field of mathematics education (PME, n.d.). Both investigations spanned the years 2000 to 2005. The aim of the ERIC and PME examinations was to identify the articles that bore reference to young children (birth to eight years), and the development of digital-numeracies or the use of new technologies.

The two research questions were:

1. What was the proportion and scope of articles on technology in the early years published in the ERIC database between 2000 and 2005?
2. What was the proportion and scope of articles on technology in the early years published in the PME proceedings between 2000 and 2005?

The first question was addressed by reporting on one aspect of a larger-scale study that examined the peer-reviewed literature on mathematics education and early childhood during the 6-year time span. The abstracts on the ERIC database on EBSCO host were the data sources for this study. Only articles from peer-reviewed journals were included because they (a) reflect the interests and values of mainstream research communities and (b) have a degree of quality control and credibility through the peer review process. In essence, the research approach consisted of identifying a data set of

articles for review from the ERIC database, limiting the data set to include only relevant articles, and ascertaining the representativeness of the literature through a thematic categorization of the articles. Categories were established that best represented the content theme of the articles. The age cohort investigated was also noted.

The second question was addressed by reviewing the contents of the annual proceedings of the PME conferences between 2000 and 2005. The PME conference proceedings were included because they (a) represent a range of international interdisciplinary research and scientific information in the psychology of mathematics education, and (b) are peer reviewed, and hence of a certain calibre in relation to quality, research significance and interest. It should be noted that some PME papers were indexed in ERIC. The research approach consisted of examining the published PME conference proceedings to identify articles relating to technology and mathematics. The data set was then further investigated to ascertain the theme of the article, and the age cohort it addressed.

RESULTS AND DISCUSSION

What was the proportion and scope of articles on technology in the early years published in the ERIC database between 2000 and 2005?

The ERIC search identified 208 articles relating to young children and mathematics which, due to multiple foci in some articles, resulted in 311 thematically-based codings. Only fourteen of the 311 codings (4.5%) focused on technology. The highest number of articles focused on mathematical concepts (36%) and the lowest number of articles focussed on problem solving (1.3%). The low proportion of articles on technology is surprising and a concern given the importance of technology in 21st century mathematics. Each of the technology articles analysed related to young children's use of computers, software, and information technology and communications during mathematical experiences within a school setting. No articles referred to digital-numeracies in the prior-to-school years. There were also no references found on new technological tools or devices to mathematical learning in the early childhood years.

What was the proportion and scope of articles on technology in the early years published in the PME proceedings between 2000 and 2005?

The total manuscripts published in PME proceedings between 2000 and 2005 was 1857. Of this figure only 145 (7.8%) peer-reviewed items contained a reference to technology and mathematics. Technology papers were represented in a plenary lecture, research forums, discussion groups, working sessions, short orals, poster presentations and research reports (see Table 1). The majority of topics in manuscripts pertaining to mathematics and technology covered topics such as software, interactive whiteboards, CAS-based algebra systems, mathematical learning aided by computers, ICT (i.e., information and communication technologies) environments and tools, pedagogy, graphics calculators, attitudes, and gender issues. The technology-themed papers in the

PME proceedings addressed various age cohorts from upper primary school children through to pre-service teachers with one exception on young children. This was a poster presentation by Hoyos (2002) titled *Computer-based mathematical games for preschool children*. Thus, over a 6-year period only one PME paper (0.05%) had an early childhood focus.

	PME 24 2000	PME 25 2001	PME 26 2002	PME 27 2003	PME 28 2004	PME 29 2005
Plenary lectures	-	-	-	-	-	1
Research forums	-	1	-	1	-	-
Discussion groups	-	1	-	-	1	1
Working sessions	Not held	1	2	2	-	-
Short oral communications	3	15	7	4	8	6
Poster presentations	1	7	6	7	5	5
Research reports	9	13	10	11	9	8
TOTAL	13	38	25	25	23	21

Table 1: Published presentations with a technology focus in PME proceedings from 2000-2006.

The analyses of ERIC and PME publications that address the early childhood years, mathematics and technology revealed significant limitations with the literature base. Proportionally, only 4.5% of ERIC articles and 0.05% of PME articles focused on this topic over a 6-year period. Additionally, only one PME paper focused specifically on the prior-to-school years, which are recognised as important for life outcomes. Moreover, none of the ERIC or the PME papers focused on new technologies. Given that today's young children are digital natives, that is "native speakers of technology, fluent in the digital language of computers, video games, and the internet" (Prensky, 2001), research is needed on the learning opportunities and demands of the breadth of technological tools and environments.

These outcomes of the review of the ERIC and PME are *not* exceptional but appear to be representative of the orientations of professional groups over time with respect to the emphasis on technology and young children. For example, 10 years ago a *technology-themed* conference by the Mathematics Education Research Group of Australasia [MERGA] (Clarkson, 1986) published 80 manuscripts, of which seven

referenced mathematics and technology. However, none of these articles pertained to young children's use of technology. Nearly a decade later, a review of research between 2000 and 2003 by the same professional community (Perry, Anthony & Diezmann, 2004) revealed an emphasis on technology in a chapter by Goos and Cretchley (2004). This chapter discussed the research on teaching and learning mathematics with computer-based technologies, and the role of computers in student learning. However, as with research published by ERIC and PME between 2000 and 2005, there was scant attention to technology and young children. Only one study on young children and technology use (Lowrie, 2002) was identified in the MERGA review (Perry et al., 2003). Lowrie's work investigated 6-year-old children's capacity to interpret and construct 3D-like images in computer environments. However, no research was reported on children's use of digital technologies in these early school years. Additionally, no research was reported on any type of technology use in the prior-to-school age range.

CONCLUSION

The review of literature published in peer-reviewed journals and PME conference proceedings identified research agendas investigating primary and secondary students', and teachers' engagement with computers, software, and within ICT environments. In the past few decades a shift in perceptions about young children's learning and mathematics in a digital age has been witnessed. However, this shift is not mirrored in the literature. The review revealed a dearth of research on digital-numeracy engagement in the early years and a paucity of research on young children's learning in the prior-to-school years. In our increasingly technological and information-based society, *mathematical proficiency* is necessary for productive participation in life and success in public and private ventures. Hence, research needs to provide adequate guidance on early childhood mathematics education in order to increase the likelihood of children's success and to develop a numerate citizenry. A substantial literature base would inform policy and practice and further validate the essential nature of early childhood mathematics and technology by providing convincing evidence about their plausible effects (Slavin, 2002). Additionally, such research would also contribute to the void of knowledge surrounding the impact of new technologies on the mathematical experiences of young children.

As technology usage and numeracy demands increase in society, it is essential that all participants are considered with special attention given to the gatekeepers of our future – the children. In order to participate and thrive in today's digital age and contribute to tomorrow's future society, individuals need to become digitally-numerate. For example, considerable mathematical knowledge and technological knowledge are required to make informed decisions about the best mobile phone and plan. The ability to make informed decisions about mathematical situations determines whether or not these individuals have first or second class economic status in society. Thus, research on children's numeracy learning and digital technologies is not only timely but necessary. Children are the pioneers of the future – what they learn, how they learn and

when they learn has 21st digital-numeracy connotations - research agendas must acknowledge and respond to this actuality if we are to fulfil the responsibility of working towards equity for all.

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TEACHING AND TEACHERS' COMPETENCE WITH ICT IN MATHEMATICS IN A COMMUNITY OF INQUIRY

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In the ICTML project the aim is to develop both mathematics teaching and the teachers' competence with ICT and to perform research on all parts of the work. The research was situated in a social cultural framework. Teachers and didacticians worked together in learning communities inquiring into approaches for teaching and how computer tools can support inquiry in both teachers and pupils work. In this connection workshops at the university college played an important part. In this paper I will report from cases of workshops and how ideas for implementing teaching with ICT were developed and discussed in the learning community.

BACKGROUND AND RATIONALE

Implementing use of ICT (Information and Communication Technology) into mathematics teaching has been a slow process. In spite of huge efforts from the Norwegian educational authorities, the use of ICT tools is still rather weak in most schools (Erstad, Kløvstad, Kristiansen, & Søbye, 2005). A lot has been achieved with general use of computers, but less in specific subjects. Hardly any activity was reported in an evaluation of teachers' implementation of the curriculum (Alseth, Breiteig, & Brekke, 2003). In my own experience, many teachers lack knowledge of how to utilise ICT tools in mathematics teaching and express need to see good examples and learn more about ways of using ICT. In the Norwegian curriculum plan, in effect from 2006, there is a demand to use "digital tools" in every school subject and specific demands are given in the plan for mathematics (KD, 2006).

The project ICT and mathematics learning (ICTML) aims to meet the challenge of investigating how ICT tools can be utilised in school mathematics and in particular how ICT can support inquiry approaches in teaching and learning. ICTML is both a development and research project, where teachers and didacticians, i.e. researchers and doctoral students at the university college, work closely together. Furthermore, the project has a close collaboration with another project, Learning Communities in mathematics, (LCM) (Jaworski, 2005), with fundamentally a common theoretical framework. Both projects are supported by The Research Council of Norway

I think of ICT tools in this context as computer software that is open and flexible not tied to specific topics or limited to pre-designed tasks. Such software provides ways of representing mathematical objects and relations and makes it possible to work on the representations. Thus ICT tools make it possible to investigate and experiment with mathematical ideas, discover patterns and relations and be stimulated in

mathematical thinking. In the project we mainly use spreadsheet, dynamic geometry and graph plotting program, and to some extent we use the Internet.

The research presented in this paper is concerned with how teachers and didacticians work together in workshops in the project and how teaching ideas and teachers' competence develop. What kind of teaching ideas were explored in the workshops? What kinds of questions were raised in the discussions? Can we find evidence of inquiry in the workshops which suggest learning is taking place? An important question is if we can see inquiry approaches to teaching emerging from the workshops.

THEORETICAL FRAMEWORK

The research and developmental work in the project is situated in a sociocultural framework with the ideas of *learning community* and *inquiry* as key concepts. The idea of learning community builds on and extends Wengers' concept community of practice (Wenger, 1998) with three modes of belonging, engagement, imagination and critical alignment as key concepts (Jaworski, 2006). The participants engage in the project activities at the university college and in schools, and develop ideas through imagination and critically align themselves with the project community by discussion and testing out ideas, and developing understanding of key concepts and the goal for the project. Furthermore, to practice inquiry is key concept in developing the learning community into an inquiry community.

Inquiry means to ask questions, investigate, acquire information, or search for knowledge. An attitude characterised by willingness to wonder, seek to understand by collaborating with others implies being active in dialogic inquiry (Wells, 2001). In the ICTML project as well as for LCM, an aim is for the participants in the projects to develop further into "inquiry as a way of being" which implies an attitude of asking questions, investigating and exploring – making inquiries into all levels in the project. This implies inquiry into mathematics, into mathematics learning and how mathematics can be represented and worked on with ICT tools.

The way ICT is used and implemented in teaching can be characterised by viewing the ICT tool as an amplifier or a reorganiser (Pea, 1987; Dörfler, 1993). The amplifier metaphor means doing the same as before, more efficiently but not fundamentally changing the objects and tasks we work on, whereas seeing ICT tools as reorganisers implies fundamental changes in objects to work on, and the way we work. For example in using a graph plotting program as an amplifier the software produces quickly the graph as the end product, whereas seen as a reorganiser the function graph itself is seen as a new object which can be manipulated either directly or by setting parameters. Use of tools as reorganiser implies a move from doing towards planning with implies work on metalevel. In a spreadsheet for example, a model can be implemented and later used for investigation trying various numbers. The calculations are left to the spreadsheet whereas the user needs to plan the model and set up connections between cells. In order to fully utilise the potential of ICT tools

such reorganising should according to Dörfler (1993), be intended and encouraged. This implies new kind of tasks and approaches to mathematics. Reorganising of cognitive processes can be seen when learners interaction with technology qualitatively transform their thinking (Goos, Galbraith, Renshaw, & Geiger, 2003). The ICT tools are not passive neutral objects, but can according to Goos et al, reshape interactions between teachers, students and technology itself.

The view of ICT tools as potential reorganisers has implications for how teaching is planned and carried through. There is need for new approaches to the work, new tasks and problems for the students to work on and perhaps new ways of working together. This can be achieved by an inquiry approach towards ICT tools, mathematics and how mathematics can be represented and worked on with the ICT tools. An inquiry attitude opens up possibility for teachers and didacticians not to know all the answers and to engage when new questions and problems arise. Teaching in this context is seen as a learning process; through inquiring into the various activities, mathematics and use of ICT, and as teaching is planned and carried through, this implies learning through the activities (Jaworski, 2006). This will be part of the development work and research in the project with the aim to meet the challenge of reorganising tasks, problems and approaches to teaching.

THE ICTML PROJECT - KEY ACTIVITIES

Four schools take part in the project and three of them also participate in the LCM project. At the start of the project in each school teachers worked together in school teams discussing teaching ideas, developing teaching and supporting each other.

The ICTML project aims to support implementation of ICT in mathematics in schools guided by an inquiry approach to teaching and learning. By asking questions, investigating and experimenting with mathematical concepts and relations the learner, whether a student, a teacher or a didactician, develops knowledge and insight in the subject area. For this we need to use ICT tools that afford this kind of activity. Inquiry on all levels of the work is crucial to the project. In the LCM project, we inquire into mathematics, into teaching mathematics and how to develop mathematics teaching (Jaworski, 2005) – and additionally for ICTML particularly we inquire into the use of ICT tools connected to all these levels.

The software itself does not create inquiry, but the way it is used, the kind of tasks and how they are presented can be crucial. For this reason design of tasks and teaching approaches are important. The design cycle can be seen as guideline for the work – to plan, act, observe, reflect and feedback to future planning (Jaworski, 2007). The teacher teams in the schools, perhaps together with a didactician, plan lessons and carry through the plan in their classes. The lessons can be observed by colleagues or didacticians and reflected over in school meetings. Feedback can be provided from this discussion or otherwise by looking at video recordings. Then another cycle can follow, revising the plan or following up by extending the teaching plan.

A team of 12 didacticians in the LCM and ICTML projects work together to develop the projects including planning for workshops and research. Two didacticians have their work dedicated to ICTML, and additionally an experienced teacher, Otto, is employed part time to support development in schools and contribute to the workshops. In addition several colleagues from the LCM project also takes part in the ICTML workshops and contribute to building the community.

At the workshops for ICTML we usually start with a session in the computer lab. The workshop will often start with some short introduction to features of the software to work on, presenting tasks or teaching ideas as examples of use. Then teachers and didacticians work in pairs or small groups on suggested tasks, investigating further ideas or inventing new applications of the software. After a short break with some refreshments, results from the work in the computer lab are brought up on a large screen, presented and discussed together with further ideas and various ways of approaching the problems. In some cases we also have reports from teachers' experiences and innovative work from their classes.

The intention is that experiences or discussions in the workshops should initiate further work in the classes. We can see the workshops as providing both competence developments for the teachers on inquiry using ICT tools and as a start of planning for teaching. In the project we see the workshops as an important activity in stimulating further development and building the inquiry community.

METHODOLOGY

The research methodology is closely connected to development in the project with the design cycle giving a framework for the activity. The research can be characterised as developmental and have roots in various other recent research methodologies, like action research, design research, learning study, lesson study (Jaworski, 2004). As the design cycle provides a model for the development it could also be characterised as a developmental cycle. Teachers are included as partners in the research, taking part in discussions and to some extent engaging in observations in classrooms with their colleagues; noticing issues that arise in the work and reflecting on experiences in the classrooms. The intention is that research in schools take place in close cooperation between teachers and didacticians.

The research is largely qualitative due to the nature of the development, aiming at describing characteristic features of inquiry approaches using ICT tools. Research is carried out on all levels in the project, including didacticians' work conceptualising and planning for workshops and other initiatives. For this reason we use video or audio to record project meetings, activities in workshops, both in plenary and in some groups, and observations in classrooms and school meetings. In addition we write field notes and reflections from observations and school meetings and collect selected tasks and students' work in computer files.

The workshops are crucial to the developmental work and for building our community. The steps in a cycle of development are not limited to the schools. The

workshop activities can provide input for the start of the developmental cycle or can play a role in the reflection and feedback part of the cycle. Another possibility is to consider the work of didacticians in a cycle of activities, for example about planning and running the workshops, where the step carrying through is the workshop.

Kennewell (2001) suggests concepts of *affordances* and *constraints* can be used to evaluate the implementation of ICT in teaching. The concept affordance was introduced by Gibson to characterise features of the objects, setting an environment which provides potential for actions (Greeno, 1994). Constraints can be seen as limitations, but are not just negative, they are rather complementary to affordances and equally necessary (Kennewell, 2001). Constraints are conditions and relationships that can provide structure and guidance. I regard these concepts helpful in analysing the ICTML work.

Due to limitations of this paper I will analyse how a few of the tasks and problems posed in two workshops challenged the participants, the outcome of the computer lab session and discussion of solutions in the plenary sessions.

WORKSHOP ON ICT AND ALGEBRA

In planning for the ICTML workshop in January 2006 the didacticians considered it valuable to follow up ideas from the previous LCM workshop which focussed on algebra. The close relation between the two projects, where most teachers take part in both, makes it possible to make such links from an LCM workshop to the next ICTML workshop. The LCM workshop dealt with functions, number patterns and how to express connections and the ICTML plan was to inquire into how ICT can support work on similar problems and what affordances and constraints ICT tools provide.

At the start, Otto gave an introduction to how the spreadsheet can be utilised to make number patterns, making connections between cells, formatting the setup to make a suitable lay out and showing how formulae can be hidden and protected.

	B	C	D	E
1				
2		Formler for geometriske figurer. Kan du finne hvilken?		
4	x	Formel 1	Formel 2	Formel 3
5	1		6	3,14
6	2		12	2
7	3		18	4,5
8	4		24	8
9	5		30	12,5
10	6		36	18
11	7		42	24,5
12	8		48	32
13	9		54	40,5
14	10		60	50
15	11		66	60,5
16	12		72	72
17	13		78	84,5
18	14		84	98
19	15		90	112,5
				706,5

Ark1 / Ark2 / Ark3 / Klar

As the participants started working on computers we observed various examples of tasks on the spreadsheet. Some made a number pattern, a number triangle, similar to the one Otto presented and others used the features of hiding and protecting formulae to make other applications.

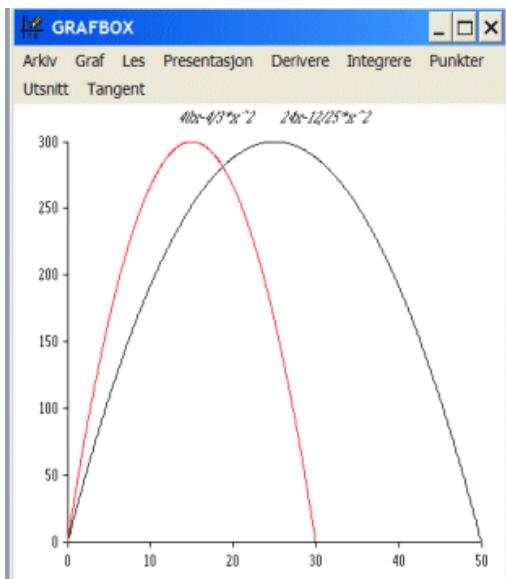
An example prepared to investigate number patterns was to reveal what formulae are hidden behind the columns. This heading gives the

task: “Formulae for geometrical figures. Can you find which one?” The spreadsheet shows three columns of numbers and the task for the students is to investigate what geometric measures are calculated in these columns. See the figure above.

When the participants later were presented with this task in plenary session, one response was “it is a rectangle”. Another said “No, it is circumference.” A question arose: Could it be both? This was discussed, and provoked further inquiry: Is it possible that area and circumference of a rectangle, when calculated will give the same number? For what rectangles will this be possible? Could the same be the case for other figures? The discussion provoked new and more general questions that can be followed by further inquiry.

Although using formulae in a spreadsheet has some relation to expressing algebraic connections, in principle, a spreadsheet is an arithmetic tool and not particularly suitable for symbolic manipulations (Dettori, Garuti, Lemut, & Netchitailova, 1995). The formulae uses cell references, not x to express variables. Two teachers wanted to challenge this and asked if it is possible to use x . During their discussion and with some input on naming cells they managed to present the same number triangle with figures and with x . In this case the teachers tried to inquire into the facilities of the spreadsheet, trying to work around the constraints of the software.

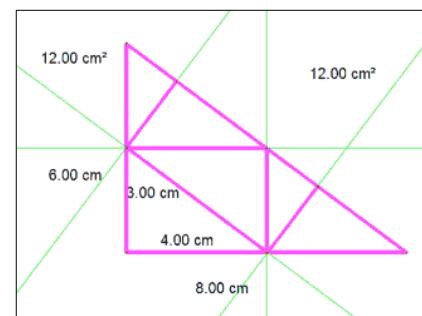
MORE THAN ONE TOOL



plotter, indicating that they were quite surprised when they saw the solution. They plotted graphs for possible placements of the rectangle. The graphs of the corresponding areas indicate that the maxima are the same. Could this be true? They showed that they had to use another method for confirming their result. They also prepared a solution in Cabri and found the same result as indicated in the figure to the right.

In the next ICTML workshop a graph plotting software was presented in the introduction. The challenge was to use more than one of the ICT tools available, spreadsheet, dynamic geometry or the graph plotter to investigate the tasks. One of the tasks presented was about making a rectangular shaped sports arena within an area determined by three roads forming a right angled triangle with the small sides 30 and 40 metre. The question is how to place the rectangle and find the maximum, with one side in the rectangle along the longest side or along the two short sides.

Various solutions were presented. A pair of didacticians presented their solution in the graph



The reflection and discussion connected to this task raised new inquiry. The task was not specifically asking for area, could we consider circumference? What if the triangular area is not right angled? Could we find the same for any triangle?

WHAT HAVE WE LEARNED FROM THE WORKSHOPS?

The workshops play a key role in developing the inquiry community related to the ICTML project. As teachers and didacticians engaged in investigations together in the computer lab, we came closer to each other and sometimes revealed our lack of knowledge of the ICT tools. This applies equally for didacticians and teachers. The experience is that this makes the roles balanced, and gives a relaxed atmosphere. It is allowed not always to know the answers, and questions that arise will lead to further inquiry. But within the project community we do have some expertise on ICT tools to provide support and suggest what further features to explore in the software. Although the teachers have some basic knowledge of spreadsheets they generally do lack knowledge of dynamic software and graph plotters.

The experiences from the summary discussions looking at various solutions from the computer lab sessions confirm that inquiry took place during the lab session and analysis shows the nature of this inquiry. The reflections carried out in plenary raised further questions to explore and investigate. Reflections provoked questions of what the software can afford and what are the constraints of the software. There were cases of utilising the constraints to support investigations, like in the task of finding what geometric measures were calculated.

On other occasions the constraints of the software challenged creativity and inquiry into the limitations of the software as in the triangle with symbolic expressions. Solving the same task with different ICT tools stimulated participants to look for connections, other alternative solutions to illuminate the tasks and develop tasks in various directions. In many cases questions were raised concerning how to extend or generalise the tasks from the computer lab, like looking at a more general triangle or look for other geometrical shapes that have the same measure of area and circumference. Other questions were about the software and what was possible. In this way inquiry into both affordances and constraints of the software took place.

CONCLUSION

The workshops, as expected, gave a huge stimulation and engagement for the work in the project and for building an inquiry community. Inquiry into ICT tools and their use for mathematics was evident in the discussion with further more general questions proposed and investigated.

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STATISTICAL INFERENCE IN TEXTBOOKS: MATHEMATICAL AND EVERYDAY CONTEXTS

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Various terms in the field of Statistical Inference and their presentation in secondary school text books are examined. A comparison of these terms in secondary school textbooks is carried out against their meanings in everyday use as well as in the mathematical context from two standard university textbooks from the field. We offer evidence that the meanings are not necessarily the same and that in some cases the definition which appears in the secondary school textbook is closer to its everyday use than to its mathematical one. Some implications for school textbook writers are derived.

THEORETICAL FRAMEWORK

Changes in mathematics secondary school curriculum, including statistics, have taken place in several Western European countries, such The Netherlands and Spain, since the late 70's. Some statistical concepts which were previously introduced during the early years at the university level are now being taught at the secondary level, i.e. confidence intervals and hypothesis testing based on the normal distribution. Some recommendations have even been made to introduce basic concepts of statistical inference during earlier schooling (NCTM, 2000), but of course without the required sophistication and formalization seen at the university levels.

Ample research into the difficulties and obstacles that students encounter when facing statistical inference have also appeared lately. Vallecillos and Batanero (1997) identified difficulties in the learning and understanding of statistical inference in the university context, especially with respect to the concepts of significance levels, parameters, and a statistic, among others, in addition to a general understanding of the logic involved in hypothesis testing. Moreno and Vallecillos (2002), researched the secondary school setting. Their study of 15 and 16 year-old students showed that students had misconceptions about statistical and carried incorrect inferences. They identify *Representativeness* as the key concept which presents the most difficulty (Kahnemann et al., 1982). Specifically, they point out that *Representativeness* is characterized by the belief that small samples must reproduce the essential characteristics of the population from which it has been taken. Students also find Hypothesis Testing to be difficult. Vallecillos (1999) indicates that students possess different ideas about exactly what a hypothesis test is. García-Alonso and García-Cruz (2003) carried out a study using a sample of (n=50) students who sat for the University Entrance Examination. They concluded that most students (86%) were unable to completely carry out those problems in the exam which dealt with Statistical Inference, even though these exercises were no different than the typical

ones that were covered in their daily classes. Previous research into this subject confirms that an inherent difficulty is present in this topic based on the logical-mathematical point of view. However our research focuses on the use of language and the effect that possible obstacles present could have on the students understanding of the topic.

Language is an important tool in the construction of mathematical knowledge, and during lower secondary school, this language is no different than that used in an everyday context, except for some very specific terms. Higher levels of “technification” and abstraction are required in upper secondary school mathematics. Hence, a greater amount of specific terms are needed in their study, which is also the case for Statistical Inference. Shuard & Rothery (1984) indicated that two contexts are present in the mathematics classroom: an everyday context and a mathematical context. An everyday context refers to carrying out typical communication found in day to day settings while a mathematical context only deals with mathematical communication. These authors categorize the terms which appear in the mathematics classroom according to these groups: (1) those terms which have the same meaning in both contexts; (2) those terms whose meaning changes from one context or the other, and (3) those terms which are only seen in a mathematical context. Students should not have any problems understanding the terms found in category (1). On the other hand, all terms found in category (3) must be defined since they are not part of a students' vocabulary. The terms found in category 2 may be difficult for students to understand. The role of textbooks was included in Discussion Group 7 at the 30th PME Conference. The need to carry out research on European textbooks was identified, and one of the specific areas covered the need to understand how textbooks are prepared and used during the learning and teaching process (Pepin, Grevholm & Straesser, 2006).

Our research into textbooks is based on the “a priori” approach. Thus we review the terms related with statistical inference and analyze them within their context. They are then classified according to the categories given by Shuard & Rothery (1984)

We wish to address the following questions: How are statistical inference terms presented in upper secondary school textbooks? Are there differences between the mathematical meaning of these terms and the ones from everyday use? Are the definitions in the textbooks correct?

OBJECTIVES AND METHODOLOGY

The are two main objectives in our research:

1. To determine if the meanings of the terms found in Statistical Inference is the same in both everyday and mathematical contexts.
2. To compare these meanings with those developed in secondary school textbooks.

Questionnaires were administered to a group of 37 secondary school teachers who were responsible for teaching the last year of an upper secondary school mathematics course. Teachers were asked to identify the publisher of the textbook they were using.

The questionnaire also revealed a very interesting fact. Approximately 30 teachers in the survey used the textbook as a means to recall and/or review the statistical concepts that their students would be working with in class. This decision emphasizes the importance of the textbook as a key element in teaching training.

We chose the four most popular textbooks according to the survey. The number in parentheses represents amount of times this book was used among the four in the study. The four publishers are: Publisher 1 (P1): Anaya (n = 24); Publisher 2 (P2): SM (n = 6); Publisher 3 (P3): Santillana (n = 4); Publisher 4 (P4): Edelvives (n = 2).

We then selected all the terms related with Statistical Inference in each publisher's version. The next step was to analyze the meaning of each term in everyday use as well as in the mathematical context. *The Diccionario de la Real Academia de la Lengua Española* (hereafter *Diccionario*) was consulted for the definitions of the terms in everyday context. Two university textbooks were used for the mathematics definitions: ME = Mendenhall (1982) and MO = Moore (2005).

Everyday context: The Spanish language is regulated by the Real Academia de la Lengua. The mission of the Real Academia is to collect all terms and new versions of their meanings which have been introduced into the language with the passing of time. The *Diccionario* is an important resource when studying the meanings of terms found and used in everyday language. *The dictionary also makes a reference to technical terms, “to introduce those words which originate from distinct fields of knowledge and also from professional activities whose current use (...) has exceeded its original meaning in another setting, and consequently has extended its use, either frequently or occasionally from either common language or in a cultural context”.*

Mathematical context: The university textbooks are used to identify the mathematical context of the terms, since they are written with the understood mission to convey the definitions of these terms and technical concepts to students. As mentioned previously, Mendenhall (1982) and Moore (2005) were used in our study.

Two definitions for each term were taken from the *Diccionario*. The first definition is the one which appears first in the *Diccionario*. The second definition is the one which is closest to the mathematical context. The definitions from the *Diccionario*, the University textbooks and the secondary school textbooks which appear in this paper have been translated into English from the Spanish versions. The English translation of the statistical inference terms are provided to guide the reader through our research methodology.

Now that the meanings of the terms from both the everyday and mathematical contexts are known, the next step is to categorize them according to the criteria given

in Shuard & Rothery (1984). If the definitions given in the *Diccionario* and the university textbooks are the same or quite similar, then the term is placed in the first category “*same meaning in both contexts*”. The third category “*specific meaning in mathematical context*” is used for those terms that are only found in university textbooks. Category 2 “*different meanings in both contexts*” is the remaining category and is used for those terms whose meanings are not the same. The final stage of our research analyzed the treatment of these terms in the textbooks from the four publishers.

RESULTS AND DISCUSSION

27 terms related to statistical inference were analyzed. Each term was grouped into its corresponding category. Each term was analyzed according to the procedure described in the previous section. Some examples by category follow.

Category 1: *Same meaning in both contexts*

Four terms were found to belong to this category: *Statistics*, *Population*, *Individual* and *Sample size*. We use the *Population* term as an example of how we carried out our study:

Term:	POPULATION	
<i>Diccionario</i>	University Textbooks	
1. Act and effect of populating. 2. Set of individuals or things subject to a statistical evaluation by means of sampling.	ME .- Set of all measurements of interest to the person who obtains the sample. MO .- An entire group of individuals of which we want to know certain information about is called a population.	
PUBLISHERS		
P1 .- “ <i>A population or universe is the set of all individuals in our study</i> ”. P2 .- “ <i>is the set of all elements that possess a specific characteristic. Populations are generally assumed to be very large</i> .” P3 .- “ <i>when a statistical study refers to a group, set or collection of elements, this collection is called the population</i> .” P4 .- “ <i>the homogeneous group of people, animals or things on which a study is to take place</i> ”.		

A comparison of the four definitions reveals how the P4 version requires homogeneity within the group. This condition is not necessary, since the purpose of the statistical study could be to study a homogeneous characteristic within the population, not a homogeneous group. Thus, we consider the inclusion of homogeneity in the definition to be misleading, and possibly causing confusion to the student.

Category 2: Different meanings in both contexts

The following terms are included in this category: *Calculated Mean, Sample, Estimation, Infer, Distribution, Probability, Representative, Risk*.

Term: SAMPLE	
<i>Diccionario</i>	University Textbooks
<ol style="list-style-type: none"> 1. Part of a product or merchandise which allows the quality of the goods to be known. 2. Part or chosen portion of a set by methods which allow it to be considered representative of it. 	<p>ME. - A sample is a subset of selected measurements from the population of interest.</p> <p>MO. - A sample is that part of the population that we are currently studying with the objective of obtaining information.</p>
PUBLISHERS	
<p>P1. - “<i>a subset drawn from the population. A study of the sample helps to infer characteristics of the entire population</i>”. “<i>However, if the sample is incorrectly chosen, (it is not representative)...</i>”</p> <p>P2. - “<i>a subset of the population</i>”. If “<i>a study is going to be reliable, it is critical that the selection of the sample be correct, so that it is clearly representative of the population</i>”.</p> <p>P3. - “<i>part of the population, carefully selected, which is subject to scientific observation as a representation of the same population. Its purpose is to obtain valid results for the entire population</i>”. In addition, “<i>a sample is considered valid when it fulfils the definition of (...) being representative</i>”</p> <p>P4. - “<i>Subset of the population</i>”. “<i>An appropriate selection</i>” should be made.</p>	

This term has equivalent definitions in both university textbooks. However, the definitions in the *Diccionario* emphasize how the subset must be representative of the complete set. This requirement does not appear in the university textbooks.

Exactly what is representative? Two definitions are found in the *Diccionario*:

1. *To recall something with words or figures that the imagination remembers.*
2. *To be the image or symbol of something, or to imitate it perfectly.*

According to this definition, if a sample is said to be representative, it is indicating that the sample must “perfectly imitate” or “be an image” of the population. These definitions lead to an incorrect idea of the term, and lead to *Representativeness*, described in Kahneman et al. (1982). *Representativeness* is defined when the student expects that small samples reflect all of the population properties. We know that its similarity with the population does not validate our sample, but instead its selection method.

It is also important to remark how the *Diccionario* indicates the need for the sample to be representative. This fact shows how an incorrect meaning in the everyday context applied to mathematics can produce an incorrect understanding of this technical term, and create barriers for students in their understanding.

Category 3: Specific meaning in mathematical context

Category 3 contains the most terms of all the groups. The list includes: *Statistic*, *Parameter*, *Random sampling*, *Sample mean*, *Population Mean*, *Confidence level*, *Standard deviation*, *Significance level*, *Inductive statistics*, *hypothetical-deductive statistic*, *margin of error*, *Normal*, *Bias*, *Efficiency*, *Sample proportion*..

Term: STATISTIC	
Diccionario	University Textbooks
1.Belonging to or related to statistics. 2.Person who exercises the statistics profession.	MO. A statistic is a number that can be calculated based on sample data without needing to use any unknown parameter. We typically usually use the term statistic to estimate an unknown parameter.
PUBLISHERS	
P1. & P4. - Undefined. P2. - “ <i>a numerical value that describes a characteristic of the sample</i> ”. P3. - “[<i>values or measures</i>] that characterize a sample”.	

The mathematical meaning of the term Statistic is not given in the *Diccionario*. Therefore it is included in the third category. When statistic is defined, it is done with references to population parameters instead of statistics. Given its importance in this concept, (statistical inference begins with this point), it is significant that some publishers exclude its definition, and that others introduce it later on in the text.

Term: CONFIDENCE LEVEL	
Diccionario	University Textbooks
Undefined	ME. - The probability that a confidence interval will include the estimated parameter. MO. A “C” confidence level represents the probability that the interval will contain the true value of a parameter in repeated sampling.
PUBLISHERS	
P1. - “ <i>Starting with a sample of size n we can estimate the value of a parameter of the population (...) Resulting in an interval in which we are confident that this parameter will be included. (...) Finding the probability that such a thing occurs. This probability is called the confidence level</i> ”. P2. - “ <i>the probability that an estimator for an interval covers the true value of the parameter which is being estimated. It is normally represented by $1 - \alpha$</i> .” P3. - “ <i>the level of confidence that we have that the population mean will belong to the interval is $1 - \alpha$</i> ”. P4. - “ <i>Calculate the two values where we expect that the searched-for parameter will be found with a certain confidence level, which we will call $1 - \alpha$, where α is the pre-determined risk level</i> ”.	

Note that publishers P2 and P3 use the probability that the estimator is included in the interval, but later on introduce new terminology to complete the definition, namely the value: $1-\alpha$. The value of α has not yet been defined. Publisher P4, however, states that α is the risk level (significance level). In other words, the definition of one term is given using another term which has not been previously defined.

Publishers P2 and P3 present an example which barely explains the meaning of a confidence level. None of the proposed activities emphasize this concept. Publisher P1 offers comments about the meaning of the concept, but Publisher P4 does not even offer an example which would help explain the meaning of a confidence level.

Recall that confidence level is a new term for secondary school students. A clear understanding of its meaning is not easy, since the concept of probability is dropped and instead levels are introduced, where the meaning of confidence appears opposite to the meaning of significance level, which represents an α error. This set of concepts and their meanings are not properly separated nor explained in a convenient manner, thereby complicating their understanding by students.

CONCLUSIONS

Our research includes the selection of specific terms related to confidence intervals and their classification according to three categories, given their meaning in two different contexts: everyday and mathematical.

The first category includes the terms found in the textbooks with the *same meaning* in both contexts. Most of the publishers provided the correct definition of the terms, although there are some examples where the definition has been significantly altered.

The second category is made up of terms with *different meanings* in both contexts. Hence, it is possible that a student can incorrectly learn the mathematical concept. We observed that most publishers use the everyday definition of the technical term, not the mathematical one.

The third category is made up of terms *only* found in the mathematical context. Some of the textbooks do not use the correct definitions. We also found examples where terms were introduced and never referred to later on in the textbook. The presentation of the material in this way can be confusing for students, or indicates that the textbook is inconsistent, since it is not continuously using terms which were introduced at a given moment.

Overall we have noticed how some definitions that appear in the textbooks do not correspond to their mathematical meaning but instead to the one in their everyday use. There are times when the definition of the term is incorrect, or it does not emphasize the key elements in the definition. Textbooks which include these errors are of great concern to students and teachers if we take into account that most professors use these textbooks to review the concepts related to Statistical Inference, and then

prepare their classes, as confirmed in the questionnaire we administered at the beginning of our study.

Incorrect definitions of technical terms, or their complete absence in the textbook has been confirmed in all four publishers. Sometime the textbook altered the meaning of the term. These differences in the definitions are not easily found, and the errors would be impossible to detect for students or anyone without a previous knowledge of statistics. We have confirmed that the terms found in the textbooks at this level are, for the most part, technical, and therefore the language used in their presentation should be consistent with their content. It is desirable that the definitions of these technical terms be correctly defined, leaving no doubt about their meaning.

In our opinion the results of our research are useful for writers of textbooks, leading to more coherent efforts by the writers, and offering a final product which is an important tool for both teacher and student alike.

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PRE-SERVICE ELEMENTARY SCHOOL TEACHERS' EXPERIENCES WITH THE PROCESS OF CREATING PROOFS

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This research examines students' participation in the process of creating proofs. In this study I adopt the communicational approach to cognition based on the work of Sfard, according to which thinking is a special case of activity of communication. Participants in this study are pre-service elementary school teachers. I encouraged students to write down the dialogue that they have with themselves while they are thinking and trying to create a proof. The results show the method of proving through writing a dialogue would be practical heuristic for involving students in the process of creating a mathematical proof.

Proof is an important part of not only mathematical practice, but also of mathematical learning and teaching (Hanna, 1989). However, research has repeatedly shown that proofs and the ability to understand and generate proofs is difficult for students in general (Hoyles, 1997) and for pre-service elementary school teachers in particular (Gholamazad, Liljedahl, and Zazkis, 2003; Martin & Harel, 1989; Simon & Blume, 1996). The main problem with many of the students is that they usually do not see the necessity of presenting an argument for a given statement (Harel, 1998). Indeed, they do not really understand what they are expected to do and why. They, mostly, consider some confirming examples as a proof (Gholamazad, et al, 2003; Martin & Harel, 1989).

Students usually work with the final product of a proving process. However, "the final proof will hardly reflect the process of generation" (Heinze, Reiss, and GroB, 2006, p. 275). The question is how can we encourage pre-service elementary school teachers to be involved in the process of creating proofs? And, what are the factors that impede pre-service elementary school teachers' participation in the process of creating proofs? In searching for a method for teaching proof, I examine the engagement in writing a dialogue as a means towards creating proofs.

THEORETICAL PERSPECTIVE

In this study, I adopt the communicational approach to cognition, based on the learning-as-participation metaphor, and conceptualisation of thinking as an instance of communication (Sfard, 2001). In this approach, learning mathematics is an initiation to a certain type of discourse: literate mathematical discourse (Sfard and Cole, 2002). Literate mathematical discourses as purposeful goals of teaching and schooling are distinguished from other types of communication through four criteria: (1) their special vocabulary, (2) their special mediating tools, (3) their discursive routines, and (4) their

particular endorsed narratives (Sfard, 2002; Ben-Yehuda, Lavy, Linchevski, and Sfard, 2005).

The communicational approach is deeply rooted in Vygotskian theory. In this approach priority is given to communicative public speech rather than inner private speech (Vygotsky, 1987). Therefore, better understanding of public discourse deepens our insight into a dialogue that one leads with oneself.

Considering the idea that thinking is a kind of communication that one has with oneself (Sfard, 2001), in this research, I encouraged students to write down the dialogue that they have with themselves while they were thinking to understand or create a proof. The main purpose of this kind of task was to satisfy the convincing aspect of a student generated proof, not only for the writer but also for the third person that might read it. Therefore, such dialogue should answer all the possible questions related to mathematical properties or arguments used in a proof. Writing down the dialogue may provide students an opportunity to reflect on their thinking process and to organize it in a convincing way. In this perspective, the dialogue can be considered as an intermediate stage between having an overview of a proof and writing a formal mathematical proof.

STUDY

Participants

Participants in this study were 83 pre-service elementary school teachers enrolled in the course “Principles of Mathematics for Teachers”, which is a core course in a teacher education program. The goal of this course is to promote the understanding of mathematical concepts and relationships. It concentrates on investigating *why* we do something in mathematical activities rather than *how* we do it.

Proof as a dialogue

To introduce the idea of writing proof through a dialogue the participants received a sample of a dialogue. The sample dialogue was between two imaginary personas: EXPLORER, the one who tries to prove the proposition, and WHYer, the one who asks all the possible questions related to the process of the proof. The main idea of designing these two personas was to consider two aspects of the character of an individual who is proving a mathematical statement.

The sample dialogue addresses the proposition: ‘The difference of an odd number and an even number is an odd number’. The main purpose of giving the sample dialogue is to show the students how many reasonable questions the proof of such simple statement might bring up. Because, in my opinion, the main problem with naïve students’ self-dialogue is that many of the reasonable questions in the process of proving are not usually ‘a question’ for them. Hence, I wanted to encourage them to pose questions. I believed writing dialogue would also cultivate the art of question posing, and after a while that would become a part of the culture of their mathematical thinking.

The participants in this study were exposed to several tasks of creating dialogue for either proving a statement or expanding a given proof for number theory propositions. Based on the purpose of the tasks, they did them in a group or individually. Giving the page limitation of this report, I focus on one of the tasks.

The task

In the given task the participants were asked to write a dialogue towards a proof for the following proposition:

Let a , b , and c be whole numbers. If a and c are relatively prime, and b and c are relatively prime then ab and c are relatively prime.

The students were asked to perform this task individually. This task was administered after they completed the section of number theory in the course. They had access to their textbooks and other knowledge sources, and the time for completing the task was not limited.

Results and analysis: students' dialogues

The source of data for the study is the students' written dialogue for proving the given proposition. A reasonable and convincing dialogue was provided by 35 of the students (almost 40%). In these dialogues, students presented convincing arguments by describing related vocabulary, choosing appropriate representations as mediators, manipulating them correctly by implementing mathematical routines, and interpreting the results by supporting them with appropriate endorsed narratives.

In this report I have organized all the dialogues created by students according to the communicational framework. For the analysis I examined the four above-mentioned components of a literate mathematical discourse in students' arguments.

The mathematical vocabulary: The data revealed that all the participants began their dialogue by recalling the definition of the words related to the proposition such as 'relatively prime numbers'. The majority of students, however, saw the necessity to situate the concept of relatively prime numbers in relation with others such as whole number, prime number, factor, and the greatest common factor. The following excerpt is part of a dialogue.

WHYer: Could you tell me what is a **whole number**?

EXPLORER: Sure, a whole number is a member of the set of positive integers and zero (0, 1, 2, 3, 4, and go on). **Integers** are defined as the set of numbers consisting of the **counting numbers** (that is, 1, 2, 3, 4, 5, \dots), their opposites (that is **negative numbers**, -1, -2, -3, \dots), and zero.

WHYer: And what is **prime number**? Can you show some examples of prime numbers?

EXPLORER: Some examples of prime numbers are 1, 3, 5, 7, and 11. A prime number is a number that cannot be divided evenly by any other number except itself and the number one.

WHYer: How many of these numbers you might have?

EXPLORER: Infinitely many.

WHYer: Fine. But what is a **relatively prime number**?

EXPLORER: Two integers are relatively prime if there is no integer greater than one that divides them both (which means that their greatest common divisor is one).

WHYer: OK. What is **greatest common divisor**?

EXPLORER: The greatest common divisor is the largest factor two numbers have in common.

Recalling all the definitions in the above excerpt shows that the writer did not take any one of the mathematical words for granted. In this situation, a written dialogue provides the students with a big picture of related vocabulary for further use in their argument. It also offers the researcher access to students' possible misuse or misunderstanding of the concepts. As can be seen in this dialogue, the student incorrectly considered 1 as a prime number, and excluded 2 from the set of prime numbers.

Mediators and routines: For presenting their arguments, students had recourse to different types of visual means that serve as communication mediators, such as numbers, verbal explanation, algebraic representation, and set theory symbols and diagrams. In accordance with the chosen mediator they implemented different form of routines.

The common aspect of all the dialogues was empirical verification of the proposition. Indeed, we can say that numerical example is the most common type of mediators for pre-service elementary teachers to communicate their ideas. Research repeatedly showed the high reliance of students on empirical verification as an acceptable proof (Gholamazad, et al, 2003; Martin & Harel, 1989). The results of this study, however, showed that only 10% of the students finished their argument in this step.

Here, I would like to distinguish between two different types of examples that students presented in their dialogue: numerical example that just verifies the proposition and a generic example. By generic example, I mean the example that tacitly expresses the process of the proof. Indeed, in such cases students compensate their lack of access to appropriate mediator by recourse to numbers. For example:

Whyer: Can you give me an example?

Explorer: Sure, let's use the numbers 15, 16, and 7. a being 15, b being 16, and c being 7. 15 is thus relatively prime with 7 and 16 is relatively prime to 7.

Whyer: How is 15 and 16 relatively prime to 7?

Explorer: When breaking down 15 the factors are 3 and 5, and when breaking down 16 they are 2^4 . These numbers are not equal to the breakdown of 7 which is 7.

Whyer: I understand! But when a and b multiplied to be ab then how does this work?

Explorer: Well when 15 and 16 are multiplied together it equals 240 but the prime factorisation still remains $2^4 \times 3$ and 5. Thus the prime factors will never equal to 7.

Whyer: Why does this always happen again?

Explorer: This is the fundamental theorem of arithmetic.

Whyer: Does this prove your statement?

Explorer: Indeed, it does! You'll remember that I stated that a and c do not have any common prime factors and b and c do not have any common prime factors. Therefore when a and b are multiplied and become ab their prime factors will not be the same as c .

As can be seen, the student drew a conclusion by choosing random relatively prime numbers and applying the Fundamental Theorem of Arithmetic to their prime decompositions. This argument, even though it is not a mathematical proof, illustrates the logic that the student has in her reasoning.

Another example of logical derivation that does not rely on mathematical formalism can be seen in responses of 9% of the participants who used the verbal explanation as a mediator and used their common sense as a routine for administering the argument. For example:

Sara 2: So none of the factors that multiply together to form a are the same as any of the numbers that multiply together to form c .

Sara 1: that's correct!

Sara 2: and none of the factors that multiply to yield b are the same as the factors that multiply together to create c .

Sara 1: exactly!

Sara 2: So how does that prove that ab is relatively prime to c ?

Sara 1: ab are combined factors of a and b because the factors of the two numbers are combined when they multiply. $10 \times 5 = 50 = 5 \times 2 \times 5 \times 1$.

Sara 2: What does that tell me?

Sara 1: Since factors of ' a ' and the factors of ' b ' have no overlap with the factors of c , this is true for the factors of ab . Therefore ab is relatively prime to c .

The majority of the participants, following empirical verification, tried to use a kind of representation in their argument, such as algebraic notations, set theory symbols, or diagrams. Around 65% of participants used algebraic representation for prime decomposition of a whole number. However, at times, their poor background in using algebraic notations and algebraic routine procedures, led almost half of them to a superficial presentation. For example, in the following dialogue the student, after an empirical verification, continues:

Aye: So, are we done?

Myself: No, because we have not proven the proposition yet, we have just seen that one example works. To prove the example, we have to go back to using letters, which represent all possibilities.

Aye: sounds good.

Myself: To start, we will express a , b , and c as a product of their primes such that $a=p_1, p_2 \dots, p_m$, $b=q_1, q_2 \dots, q_n$ and $c=r_1, r_2 \dots, r_q$. Therefore $ab=p_1q_1, p_2, q_2 \dots, p_mq_n$. Since the Fundamental Theorem of Arithmetic states that each composite number can be expressed as the product of primes in exactly one way and

we know that $ab=pmqn$ and $c=rq$, we can state that ab and c are relatively prime.

Aye: And that was all we needed for proof?

Myself: Yes!

Endorsed narrative: The endorsed narratives are the production of discursive activities and mathematical routines (Ben-Yehuda et al, 2005). In a deductive reasoning all the claims should result from legitimate routines or be supported by the narratives, such as definitions, postulates, and theorems that are accepted by the mathematical community as true. Therefore, the process of proving can be also considered as producing a chain of endorsed narratives that leads the argument to the desired conclusion.

The Fundamental Theorem of Arithmetic is an endorsed narrative that can support and guide the whole process of a proof for the given task. In several dialogues, however, it was observed that students made a claim about the prime factorisation of whole numbers without referring to this theorem. The following excerpt is an example of such presentations.

Me: Right. Let a , b , and c be expressed as their product of primes as follows:

$$a = p_1 p_2 \cdots p_t, \quad b = q_1 q_2 \cdots q_f, \quad c = r_1 r_2 \cdots r_y$$

My head: What does this tell us?

Me: This shows that a and c are relatively prime because their products of primes do not have any common primes. It also shows that b and c are relatively prime because they too do not have any common primes.

My head: But then how do we know that ab and c are relatively prime?

Me: We know this because $ab = (p_1 p_2 \cdots p_t)(q_1 q_2 \cdots q_f)$ and these do not have any common primes with c , $(r_1 r_2 \cdots r_y)$. Therefore ab and c are also relatively prime because their greatest common factor is 1.

My head: So can we now say that we proved the statement?

Me: Yes, by expressing a , b , and c as their products of primes we have shown that if a , b , and c are whole numbers and a and c are relatively prime and b and c are relatively prime, then ab and c are also relatively prime because their greatest common factor is 1.

In this dialogue, the students made the conclusion simply based on the appearance of the selected letters for prime factorisation of a , b , and c .

DISCUSSION AND CONCLUSION

In this report, I have introduced the notion of dialogue as a tool for involving pre-service elementary school teachers in the process of creating a proof. Here two legitimate questions might emerge: What kind of dialogue? How could the dialogue be helpful? As I mentioned above, by dialogue I mean a self-dialogue or a conversation that a person has with oneself while s/he is thinking. Therefore, writing the dialogue provides students with an opportunity to reflect on their thinking process, to correct the mistakes and fill the gaps in their argument, and to organize it in the form of a convincing mathematical discourse. It also provides a good source of students'

discourses for educators and researchers. Indeed, it offers an opportunity for educators to examine students' arguments, and by posing more appropriate questions lead the students to refine and strengthen their arguments.

The benefit of writing dialogue is that it encourages students to 'explain *why* and *how* to do' instead of just doing. This is what Schoenfeld (1994) calls mathematical culture, where discourse, thinking things through, and convincing are important parts of students engagement with mathematics. He believes in such environment "proofs would be seen as a natural part of their mathematics (why is this true? It's because...) rather than as an artificial imposition" (p. 76).

Having clear understanding of the mathematical vocabulary and using it appropriately is very important component of a mathematical discourse. The study demonstrates that explaining mathematical words is a common part of the dialogues. The majority of the students have posed questions regarding the explanation and application of words or symbols and they answered them. That is to say, writing dialogue led them to make all the related definitions available and engaged them in the process of the proof.

One of the common questions in all the dialogues was (in different phrases): "Does a numerical example prove the proposition?" 90% of the participants answered 'no', which is a promising result of the study. They continued their argument in general by using different form of mediators. The result showed writing dialogue was useful for changing students' attitude toward empirical proof that was a cognitive obstacle in their understanding of proof. Indeed, it provided them with a chance to face the conflict of whether some limited number of examples can guarantee the validity of a statement in general.

However, poor access to appropriate mediators, especially in the form of algebraic representation and poor skill in discursive routines, are still the main challenge for pre-service elementary school teachers to present their argument mathematically. The results revealed that the main difficulty that students experienced in creating a proof is that they do not know how to communicate their idea mathematically. Nevertheless, I would like to acknowledge students' ability to implement, and in some cases even invent, different kinds of mediators to present their idea.

Indeed, the paradigm of dialogues provided pre-service elementary school teachers with a flexible environment where they could cultivate their reasoning in the form of a literate mathematical discourse. Also, having a close look at students discourses, presented in the form of a self-dialogue, under the lens of communicational framework provided the researcher with indicators for recognising the factors that impeded students in the process of creating a proof.

This study is a part of an ongoing research on using dialogue for teaching proof. Based on the results, I believe that creating a dialogue is a helpful intermediate stage towards writing a proof. Further research will explore additional implementation of this method.

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INTERNATIONAL SURVEY OF HIGH SCHOOL STUDENTS' UNDERSTANDING OF KEY CONCEPTS OF LINEARITY

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Developing students' understanding of key concepts of linearity and their various representations is a major focus of first courses in algebra for secondary school students. Students' difficulties with problems involving linearity on Massachusetts high stakes tests motivated an in-depth study of the nature and causes of these difficulties. The study population expanded to include more than 4000 students in Algebra I in the USA, Korea and Israel. A written test and clinical interviews were designed and administered to the population. Clinical interviews were conducted to gain greater insight into solution methods to selected test items. The study was an activity of the Focus on Mathematics project funded by the National Science Foundation (NSF/EHR-0314692).

INTRODUCTION

Several attributes of a line, such as slope, y-intercept and equation of a line are considered to be core concepts in the secondary school mathematics curriculum. They are understood to have simple internal structure and they are taken to be the “foundations” out of which more complex concepts such as function are developed, or related to, as in the case of the relationship between the concepts of derivative and slope. The NCTM *Principles and Standards for School Mathematics* state that “Students should be able to use equations of the form $y = mx + b$ to represent linear relationships, and they should know how the values of the slope (m) and the y-intercept (b) affect the line.” (NCTM, 2000, p.226-7). The study described here demonstrates that we are far from achieving this goal, not only in the U.S. but also internationally.

MOTIVATION FOR THE STUDY

Increasingly, school districts throughout the United States are calling for algebra for all secondary students (NCTM, 2000; National Mathematics Advisory Panel: Strengthening Math Education Through Research, 2006). Concomitant with the call are articles reporting huge student failure rates in Algebra I. This national problem was particularly noticeable in Massachusetts with secondary school students' poor performance on linearity items on the Massachusetts Comprehensive Assessment System (MCAS) tests (Massachusetts DOE, 2000 – 2006). Identifying students' difficulties with linearity concepts and skills and developing strategies for addressing those difficulties was the work of the Curriculum Review Committee (CRC), a committee of the Focus on Mathematics Project. In 2004, the CRC consisted of mathematics and mathematics education faculty from Boston University and mathematics curriculum coordinators from five school districts in the Boston

metropolitan area. In 2005, the committee was joined by mathematics education faculty from Korea and Israel.

To begin their work, the CRC “unpacked” each MCAS item dealing with linearity to identify the type of displays and formats that students had to interpret, and the mathematical concepts, skills and reasoning methods students were required to bring to bear to solve the problems. The CRC also speculated about reasons for students’ difficulties. To verify conjectures about the nature of student difficulties, the CRC designed, conducted and video-taped interviews of students solving selected MCAS problems and describing their thinking. Interviews were analysed and “real” difficulties were compared with the CRC’s speculations. The CRC was on target about 60% of the time; many student difficulties were totally unsuspected. To gain more information about the difficulties, the CRC developed a written assessment instrument as well as a clinical interview to probe students’ difficulties with selected items.

TEST INSTRUMENTS

To be successful with algebra, expectations are that students understand functions both as input/output rules and know how they are represented in graphs and tables; understand that linear functions are characterized by constant slope, and can plot graphs of linear functions given in the general $ax + by = c$ form; can relate the slope of a linear function to the speed of an object represented in a time vs. distance graph; and understand the significance of the sign of the slope of a linear function in the previous context. It was this set of “understandings” that formed the basis for the design of the Mini-Assessment Test (MAT).

The Mini-Assessment Test (MAT) consists of seven items that use the same formats (essay, short answer, and multiple-choice) as the MCAS tests. For each item In the MAT, scoring directions were developed to take note of specific types of errors.

Essay Response (1 item): Given coordinates of a point and the equation of a line, students determine if the point is on the line and describe their decision-making process. (#1 in the test).

Short Answer (3 Items): 1) Given an equation of a line with a negative slope, students create a table of values (coordinates) of points on the line. (#2 in the test). 2) Given a graph of a line, students identify the slope of the line. (#4 in the test). 3) Given a distance-time graph, students identify the part (one of three) of the graph that represents the car moving slowest (PartA); the slope of another part of the graph (Part B); and the car’s speed in that other part of the graph (Part C). (#7 in the test).

Multiple-Choice (3 Items): 1) Given a table of (x, y) values representing points on a line, students identify one of four graphs that contains all points. (#3 in the test). 2) Given a table of values showing a relationship between number of weeks and number of cars sold, students identify one of four linear equations that represents the relationship. (#5 in the test). 3) Given a linear equation that is not in slope-intercept

form, students identify one of the five possibilities for the value of the y-intercept. (#6 in the test).

SUBJECTS

In the US, one week after students completed the MCAS tests in May, 2005, classroom teachers in the five districts administered the MAT to 752 Grade 8 students and scored the tests.

In Korea, 405 Grade 8 students from five schools in Seoul similar in socio-economic status to the US schools took the Korean version of the MAT in October 2005.

In Israel, since linearity is introduced to Grade 9 students, 575 Grade 9 students from seven representative schools took the Israeli version of the MAT in April 2006.

RESULTS FROM 3 COUNTRIES

Since the test conditions and the grade levels of the subjects were not equivalent across countries, data were analysed to identify trends.

The results of % correct by country are shown in Figure 1.

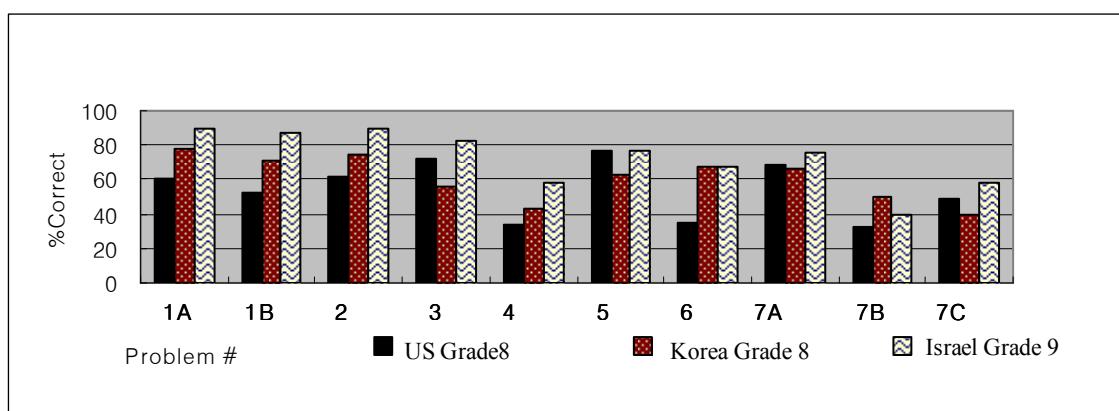


Figure 1: FOM Algebra Test Results - % Correct

Findings revealed that across the three countries, students have minimal understanding of two major topics: points on a line and slope.

With regard to points on a line, many students didn't know or weren't sure that: 1) coordinates of points on a graph of a line satisfy the equation of the line, and 2) coordinates of points on a line that are presented in tabular form satisfy the equation for the line, and when plotted, produce a graph of the line. In a study conducted by Schoenfeld, Smith and Arcavi (1993) similar results were found. They asserted that this misunderstanding is caused by the absence of what they called the "Cartesian Connection."

With regard to slope, students in all countries demonstrated maximum difficulty determining if lines shown in the coordinate plane have positive or negative slopes. Particular difficulty was noted when lines with positive slopes were pictured in the

third quadrant of the coordinate plane. As can be seen in what follow, in many cases the difficulties encountered by the students are in accord with Acuna's (2001) claim that the visual information in the graph is an important part of graph comprehension. The students use their own "gestalt" relation that takes place on the visual identification level despite previous training or knowledge of the definition of slope. Another misconception demonstrated by the students is in line with Schoenfeld et al's (1993) claim that "students can treat the algebraic and graphical representational domains as though they are essentially independent."

In the remainder of this paper, discussion will focus on the concept of slope.

Difficulties with the concept of slope

Many students had difficulty identifying slopes of lines from graphs of the lines (Problem 4 and 7B) and recognizing the relationship between slope and speed in a time versus distance graph in Problem 7, parts A and C. From our qualitative analysis and comparison of subjects' incorrect answers to the two problems, several error patterns were revealed:

- Some students did not attend to the direction of the x value change. For example, students counted the y value change from the point on the Y axis, and identified a negative slope $(-3/1)$ in Problem 3 and a positive slope $(90/3=30)$ (90 is wrong) in Problem 7B. (See Figure 2)
- Students did not notice the relation between the direction of a line and the sign of its slope and only considered the changes of x and y values. They presented positive and negative slopes for lines slanting in the same direction, seemingly unconcerned with this inconsistency. As shown in Figure 3, one student produced a table of values and calculated the y difference as 3 and gave the answer as -3.
- Students' calculations were often based on their visual judgement. For example, as shown in Figures 2 and 3, students used the estimated coordinate $(1, 30)$ instead of the correct lattice point $(3, 120)$ to calculate the slope of R in Problem 7B, and gave an answer of 30.
- Students did not take into consideration the scale factor on the Y axis (each unit represented 30 miles) in calculating slope. Many students answered $4/3$ to Problem 7B instead of 4×30 , or $120/3$ as shown in Figure 4
- From different results and different solution processes for Problem 7B and 7C, it is clear that many students did not realize the relationship between slope and speed in the distance-time graph. (See Figures 2, 3, and 4).
- Some students confused linearity with proportionality. For example, S_4 wrote "Since the slope is (increment of y) / (increment of x), and x value and y value of point A are -3 and -5 respectively, the slope is $5/3$." The student presented the correct answer for Problem 7B: "In this graph, slope is speed, and speed is (distance)/(time). Therefore $120/3=40$." (Figure 5)

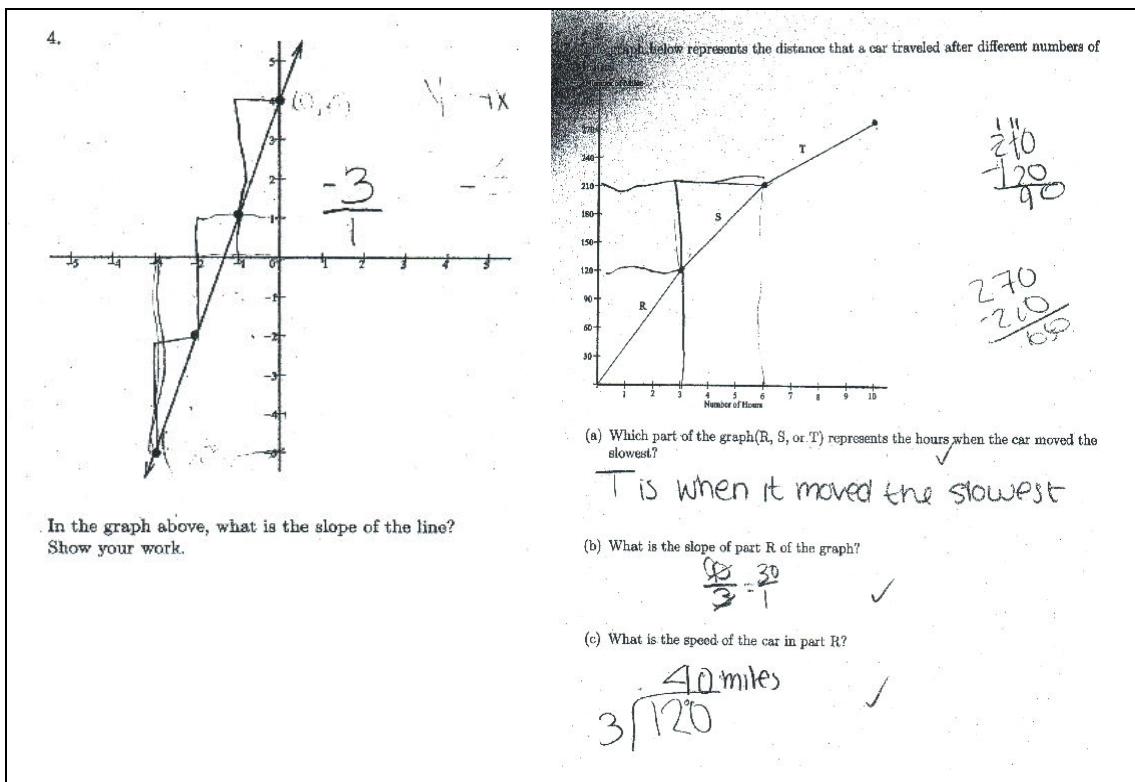


Figure 2: Student's Response-S1 to Problems # 4 and # 7B

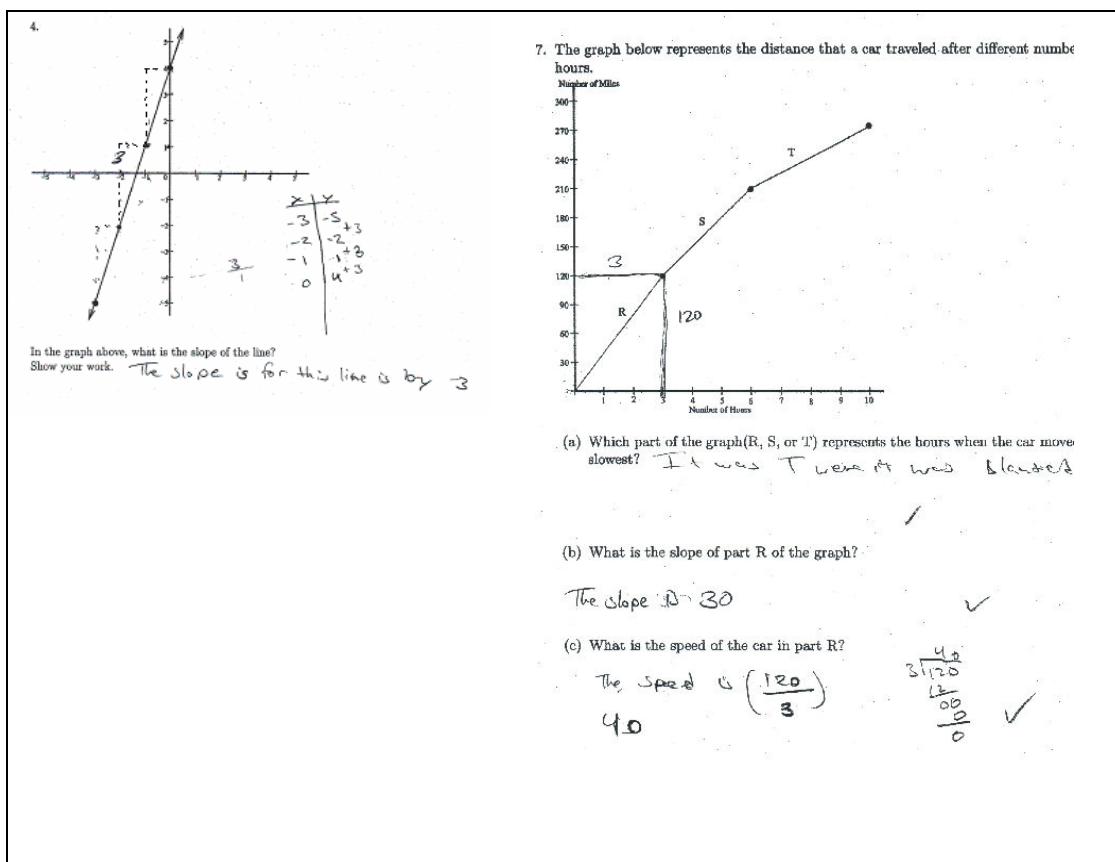


Figure 3: Student's Response-S2 to problems # 4 and # 7B

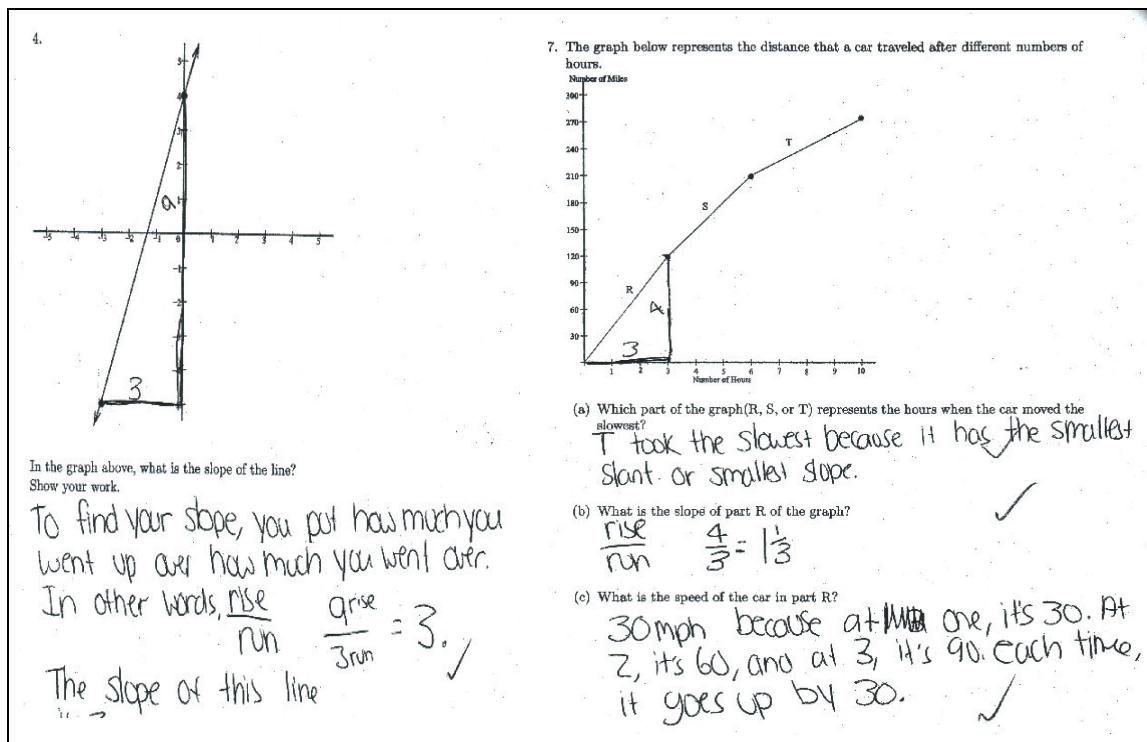


Figure 4: Student's Sample Response-S3 to Problem # 4 and #7B

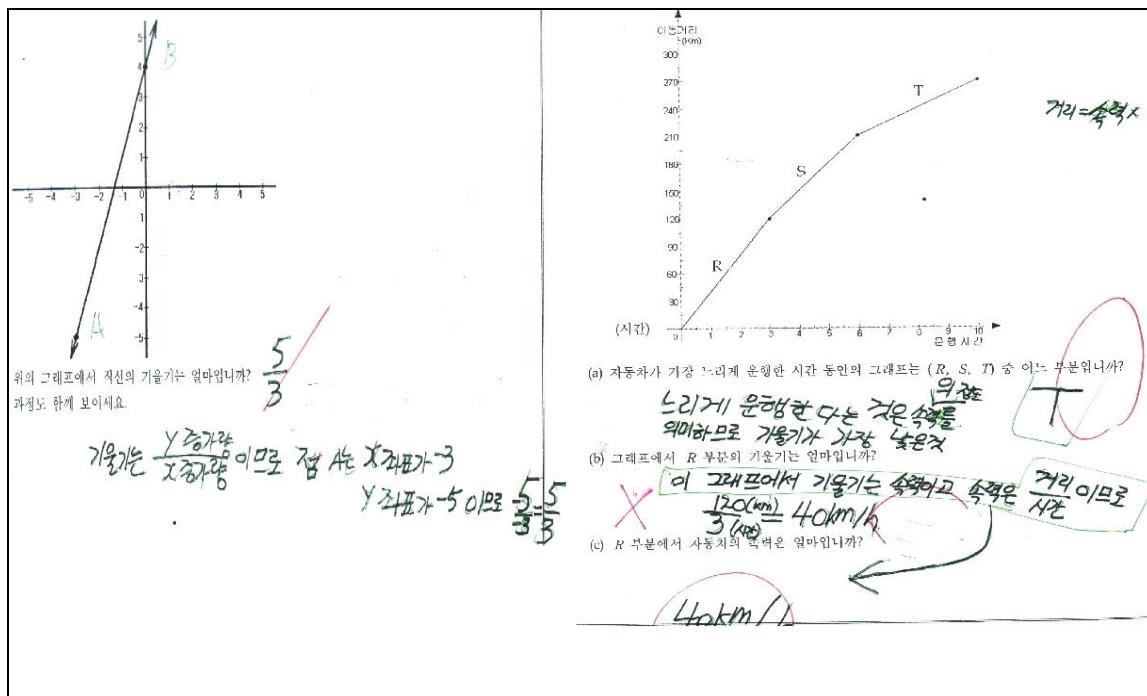


Figure 5: Student's Sample Response-S4 to Problem # 4 and #7B

To validate suspicions about the nature of the errors on the test, the CRC conducted taped interviews of grade 8 students solving selected MAT problems; one to determine the slope of a line from its graph and the other to interpret slope in an application problem. The difficulties cited above were confirmed in the interviews.

The CRC next approached the problem of identifying factors that might be contributing to student difficulties, including problem format, the grade level of the students, and the type of instructional program. For example, in the original problem #4, no grid lines were shown, and although the axes had hash marks, the scales were not indicated. The CRC believed that the lack of grid lines and the unmarked axes presented a new situation for the students, one for which they were unprepared. To check out this hypothesis, three forms of that problem were developed and administered to 1000 grade 8 students. One of the forms was the original presentation of the problem, a second showed grid lines, and a third showed grid lines and scales on the axes. Scores were analyzed and no significant difference by format was found. Similar results of no significant difference were obtained for the factors of grade level and instructional program.

Aptitude-Treatment Interactions (ATI)

With regard to slope, two kinds of Aptitude-Treatment Interactions (ATI) were revealed among groups: 1) In Problem 4 and Problem 7B and 2) in Problem 7B and 7C.

One type of ATI was in difficulty identifying slopes of lines in Problem 4 and in Problem 7B: 1) US students showed the same levels of difficulty with Problem 4 as with Problem 7B, 2) Korean students showed greater difficulty with Problem 4, and 3) Israeli students showed had greater difficulty with Problem 7B.

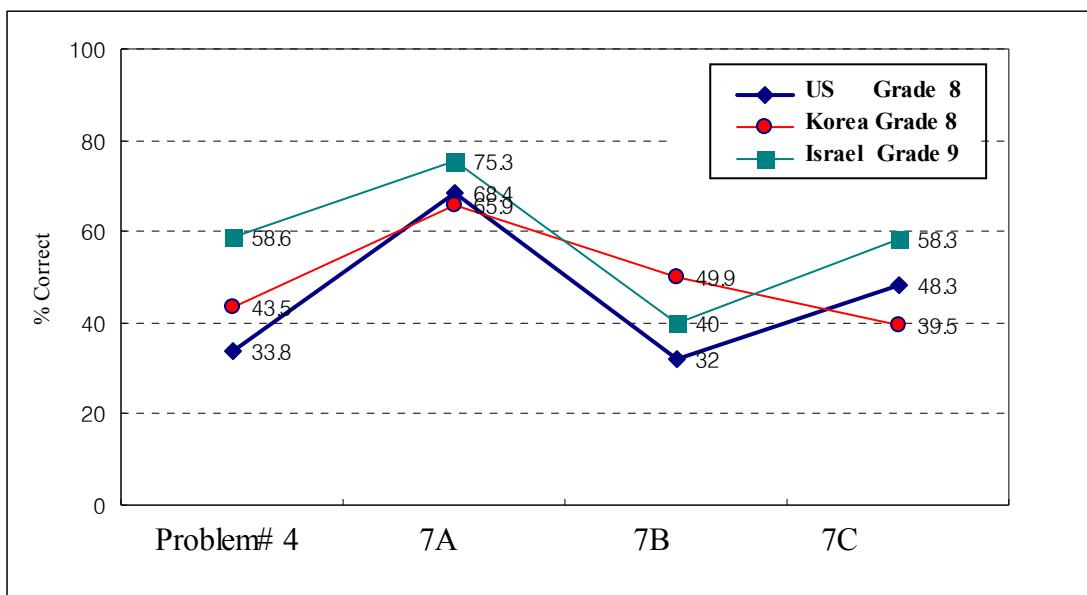


Figure 6: FOM Results on Problem 4, 7A, 7B, and 7C

As can be seen in Figure 6, additional ATI were shown in difficulty calculating slope and speed from the distance-time graph: 1) US and Israeli students showed greater difficulty with calculating slope in Problem 7B than with calculating the speed of a car from the distance-time graph in Problem 7C, but 2) Korean students showed greater

difficulty calculating the speed of a car from the distance-time graph (Problem 7C) than with calculating the slope of a line (Problem 7B).

ATI might reflect the curriculum and instructional program for linearity in each country, as well as the methods and sequences of introducing linearity and the extent of connecting concepts to their applications.

CONCLUSION

Our results are in agreement with the assertion of Schoenfeld et al (1993) “that some aspects of the domain that we take to be trivial are major stumbling blocks for students”. Knowing what the underlying skills and perspectives actually are, has implications for mathematics curricula and can serve as a guide to developing curricula. In fact, the CRC has started this year to look at the elementary school mathematics curriculum with the intent of inserting introductory material that will enhance students’ understanding of the important key concepts of linearity. Another recommendation is related to strategies used to teach those concepts. To enable students to build deep and meaningful understanding of the key concepts of linearity, it is recommended that teachers use the spiral method and devote much more time to teaching and systematically reviewing concepts of slope, y-intercept and the connection between the algebraic and graphical representation of a line.

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MATHEMATICAL BELIEFS IN PICTURES AND WORDS SEEN THROUGH “MULTIPLE EYES”

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When students create a picture on what mathematics is for them and explain their works in a text, this data can be examined regarding mathematical beliefs. A research design for investigating pictures and texts on mathematical beliefs is carried out by pre-service teachers. They work as raters who classify the students' works according to established categories of instrumentalist view, Platonist view, and problem-solving view. The agreement of the ratings of the students' works according to these categories turns out to be satisfactory. Correlations between the rated categories and certain criteria for the classification of the works are considered.

MOTIVATION

The research field of beliefs has a long tradition and is well established (Leder, Pehkonen & Törner, 2002). However, researchers mainly employ questionnaires and interviews to investigate mathematical beliefs (Hannula, Maijala & Pehkonen, 2004; Op 't Eynde & De Corte, 2003). Even if this research has provided meaningful insights in this complex construct, the difficulties and constraints regarding these methods have also been largely discussed. Recently, we have developed and explored a research design to investigate student beliefs about mathematics by using other than the above mentioned methods to collect data (Halverscheid & Rolka, 2006; Rolka & Halverscheid, 2006). We first asked the students to express their views on mathematics by drawing a picture and then to explain their picture by writing a text. In case that the information based on these two data sources remained unclear, we additionally conducted an interview with the students in question. This procedure to employ “multiple sources of evidence” in the sense of triangulating the methods is emphasized by Schoenfeld (2002, p. 463) as one source of trustworthiness of scientific results. The analysis of this rich body of data was guided by Ernest's (1989; 1991) categories to describe mathematical beliefs.

However, as continuing the analysis with a considerably large amount of pictures, the question arises how objective the interpretations of the pictures are. In fact, this problem was brought up by participants in the presentation of the method to use pictures and texts at PME 30 in Prague. To explore this question we developed a design that follows the principle “multiple eyes on the same data” (Schoenfeld, 2002, p. 463) - another criterion that Schoenfeld considers necessary for the trustworthiness of scientific results. This means that several analysts were trained in order to check the independency of the interpretation. This paper deals with presenting our procedure.

THEORETICAL FRAMEWORK

Beliefs are often considered as a construct composed of different categories (Dionne, 1984; Ernest, 1989; 1991; Grigutsch, Raatz & Törner, 1998). Even if different researchers use different notions, the meaning of the categories is more or less the same. We employ the notions of Ernest and use this section to recall what is understood by them. Further, we give an explanation what the categories mean in the case of pictures and texts. Hence, we extended an existing theoretical framework that was mainly created and justified for questionnaires and interviews to other methods, namely pictures and texts. A detailed description and example pictures that illustrate the categories can be found in Halverscheid and Rolka (2006) as well as in Rolka and Halverscheid (2006).

In the instrumentalist view, mathematics is seen as a useful but unrelated collection of facts, rules, formulae, skills and procedures. Transferred to pictures and texts, one important feature is the disconnectedness of mathematical objects. There are no connections made between different ideas mentioned in the works. The pictures often consist of an unrelated enumeration of objects that are not ordered to a mathematical statement.

In the Platonist view, mathematics is characterized as a static but unified body of knowledge where interconnecting structures and truths play an important role. Compared to the first category, the students here try to connect different elements. One student, for example, refers to Pythagoras' theorem as geometry and states that "it is fascinating that one can make geometry with algebra". However, students in this category do not create mathematics. The view on mathematics remains static and related to historical truths and links between them.

In the problem-solving view, mathematics is considered as a dynamic and continually expanding field in which creative and constructive processes are of central relevance. Students in this category produce mathematics with things from their environment that they use as starting point for mathematical thoughts and activities.

The following criteria were worked out as typical for the instrumentalist view, Platonist view and problem-solving view (Halverscheid & Rolka, 2006; Rolka & Halverscheid, 2006).

Instrumentalist view:

- Disconnectedness of the objects
- No dynamical view on mathematics
- No mathematical statement
- Usefulness of mathematics

Platonist view:

- Mathematical topics are represented
- Elements are connected
- No story is told, no mathematical activities are shown
- Historical interest, famous mathematicians

Problem-solving view:

- Mathematical activities with painted objects
- Mathematics as a dynamic field
- Students participate actively in the creation of mathematics

These criteria were obtained by independent interpretations of works by the two authors. If just one of these criteria is met, this does not necessarily imply that the according category is appropriate. The individual character of each work makes it only possible to give a catalogue of typical properties; the classification task remains interpretative. The rating procedure should also give some idea of whether there is a correlation between the classification as the established categories and the criteria which are thought to be indicative for mathematical beliefs. Methodologically, it would be important to know if there is a correlation between the criteria and the classification according to categories of mathematical beliefs.

METHODOLOGY

As already mentioned, our research was guided by the question to what extend the interpretation of the students' products are independent of the judgments of different raters. According to Schoenfeld (2002), it is therefore common to compute the interrater reliability "to identify the degree to which independent researchers assign the same coding to a body of data" (p. 463). In order to check the interrater reliability, we first presented 22 pictures created by fifth-graders to a group of 16 pre-service teachers. They were asked to answer the following prompts:

- What is characteristic for the picture?
- What is the mathematical idea in the picture?
- What would you ask the artist in order to understand the picture?
- Do you have any additional remarks?

For every picture, 150 seconds of consideration were given. Subsequently, each pre-service teacher had to put the pictures that – in his or her eyes – seem to express a

related idea into groups. In an essay, every group was described in detail and for every of the 22 pictures, it had to be explained why it was classified in the respective group.

Then, the participants took part in training in order to prepare them for the interpretation of the pictures and texts according to Ernest's categories. They were shown example pictures of each category and received the typical features listed above which help to identify the category. This training covered a whole lesson. In the next lesson, these features were briefly recalled and we then presented six pictures and texts out of the 22 for categorization to the students.

FINDINGS

Participants' notes on pictures without text

The participants' comments on the pictures were made at a stage when the text had still been unknown to them. Their remarks show that it was not that difficult in many cases to recognize what objects are illustrated. In most cases, the mathematical content of the picture is for the observers restricted to the symbols and objects shown.

Since, at that moment, the pre-service teachers had not taken part in the training how to classify the objects according to the categories of mathematical beliefs, it was our intention to see which questions they would like to get answered by the students.

The pre-service teachers were given deliberately a lot of freedom to make remarks on the works. This makes it difficult to relate the remarks to the categories established later. However, they tend to have formulated more questions on works, which are rated later as belonging to Platonist view, or problem-solving view than to those classified as instrumentalist view. Often, concerning those pictures no questions are asked at all. Furthermore, many questions look for the connection between certain elements on the picture.

Occasionally, as a feature, which is not represented in the established categories, affects play a role. "Is mathematics fun?" or "Do you like calculating?" is asked; another pre-service teacher notes that a certain picture describes mathematics within nature and suggests interviewing the student to find out whether this is related to the student's feelings on mathematics. The participants also broach the issue of links between mathematics as a science and at school.

Essays on categorization of works

In this step, the pre-service teachers were given both pictures and texts to work out categories for classifying the 22 works.

The majority of pre-service teachers use categories describing objects or topics which are quite explicitly contained in the pictures. The most frequently mentioned categories involve mathematical objects (like numbers or geometric figures, which are often taken as separate categories), the usefulness of mathematics (mathematics in everyday life or mathematics in professions), relations to other sciences or the history of

mathematics and science (including examples of mathematicians and scientists like Albert Einstein).

Other aspects of categorizing present in the pre-service teachers' descriptions refer to social relations and mathematics: Mathematics at school, philosophical approaches to mathematics, and mathematics in relation to people. Affective components are used in two cases, where personal feelings about mathematics and the fun or frustration while learning mathematics are discussed. Three pre-service teachers consider the role of the students to distinguish how they look at mathematics: They take categories like: Associations with mathematics, mathematics as part of a personal philosophy, mathematics in the lifespan.

Data analysis of the rating

The raters classified each work including picture and text according to the established categories of instrumentalist view, Platonist view, and problem-solving view. The interrater reliability can be measured with Cohen's kappa (Cohen, 1960). It was only possible to consider 10 raters who participated in all training and rating units. Cohen's coefficient is determined for each pair of raters, and the median of all coefficients is given here. The median is $\kappa = 0.71$, which is considered to indicate a good degree of reliability (Fleiss & Cohen, 1973).

In the classification in Halverscheid and Rolka (2006), criteria are elaborated as indicative for instrumentalist, Platonist or problem-solving view. Six criteria were given to the raters because these are considered distinctive for the classification. We list these criteria below; in brackets Cohen's kappa (considered in each case separately only for the according criterion) is given for the reliability of this criterion.

- Work shows several, rather not connected elements ($\kappa = 0.4$)
- Usefulness of mathematics ($\kappa = 0.33$)
- Mathematical activities with painted objects ($\kappa = 0.18$)
- Description of mathematics as a dynamic field ($\kappa = 0.67$)
- Elements are connected ($\kappa = 0.31$)
- Interest in (historical or theoretical) development of mathematics ($\kappa = 0.33$)

If a work appears as a sample of rather disconnected elements, this is regarded, according to Halverscheid & Rolka (2006), as an indication of an instrumentalist view. In fact the correlation of the observation that the work shows many rather disconnected elements and of the classification of the work as instrumentalist is 0.52.

If the utility or usefulness of mathematics is stressed, this is considered as an indication for an instrumentalist view. The correlation of these categories is only weak: 0.13. It seems that the criterion for disconnectedness is much more indicative for the instrumentalist view than that of usefulness of mathematics.

In the catalogue of categories a difference was made between the disconnectedness of elements and the property that a context of mathematical elements is depicted. These categories were treated as different categories, and not only as opposite elements which negate each other. The category describing that a mathematical context is constructed correlates positively with problem solving view (0,23) and with Platonist view (0,11), whereas it correlates negatively with instrumentalist view (-0,33).

DISCUSSION

Interpreting pictures without texts

The raters classified each work including picture and text according to the established categories. The first step of the rating shows how difficult it is to get reliable information from pictures alone. It is often clear which objects are presented, but links between certain elements and other information are hard to extract. The affective side of mathematics and a discussion of existing or missing links between mathematics at school and in the rest of the world come up already by just looking at the pictures. This could be an indication that the role of affects for mathematical beliefs could be examined more closely. It also would be interesting to understand better the differences between beliefs of mathematics and of the learning of mathematics.

Different approaches to classifications

Ernest's well-established classification has been examined intensively in many situations. A different question is whether the task to draw a picture and to explain it in a text can be extended to further research questions. Categories that raise the topic of different social roles of (the learning of) mathematics and of affective components of mathematics appear worthwhile considering. The affective side of the works plays a role when the works at different grades are considered (Rolka & Halverscheid, 2006).

Interestingly, several criteria for instrumentalist view are considered by most of the pre-service teachers. For the Platonist view, the historical category is often found. However, problem-solving view and Platonist view were not distinguished as long as they were not presented. This is another indication that Platonist view and problem-solving view are difficult to distinguish. One reason why it did not appear in the pre-service teachers' classifications could be the fact that both appear quite rare and that the instrumentalist view is clearly dominant.

Implications of the rating procedure

The results on the agreement of the ratings indicate a good interrater reliability of the interpretation of the works according to the established categories by Ernest. Since the rater training of the pre-service teachers did not take too much time, it could be hoped that the method of using pictures and texts simultaneously for investigating mathematical beliefs could be extended to a method which is not only applicable by specialists.

The criteria suggested in Halverscheid and Rolka (2006a) show a lower degree of reliability. Still, the interrater coefficients reach from weakly positive to positive. The criteria can be understood as a catalogue which characterizes the categories, but there are no automatic implications.

For an interpretative task, it is also not surprising that the correlations between the criteria and the classification indicate a lower degree of correspondence. The disconnectedness of presented elements appears as the most significant indicator for an instrumentalist view. Cohen's kappa for disconnectedness indicates, however, that it might be difficult to decide whether the criterion is met.

Although the methods of data collection applied here differ very much, the quality of the rating procedure might have improved by going to three steps – similarly to the students who had created their works in several steps, too.

OUTLOOK

The method of training the raters and evaluating their classifications is useful to check the consistency of the methodology. The agreement of the ratings of the students' works according to the categories of instrumentalist view, Platonist view, and problem-solving view turns out to be satisfactory. For an interpretative task, this consistency of the method is an important step for the implementation of the theory. It allows classifying works by a trained rater.

A very different question is whether the established categories describe the works extensively. For this question, more open, interpretative methods have to be considered. The classifications of the works by pre-service teachers indicate that affective and social aspects could be rich sources for further analyses.

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